Faculty of Philosophy

# Formal Logic

## Lectures 12–13

Peter Smith

# Outline

- Single-signed trees

- Unsigned trees

- Extending trees to cover conditionals

- Finally, the rules for biconditionals

# Summary so far

To operate the ‘working backwards’ method for seeing whether an inference is tautologically valid we . . .

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3. If a branch contains a contradictory assigment of values to wffs, close it off with an absurdity marker – this branch can’t represent a possible way of making the premisses true and conclusion false.
4. If all branches eventually get closed off with an absurdity marker, then the assumption that the inference is invalid has lead to unavoidable absurdity and the inference is valid.
5. If a branch remains left open when we have unpacked the implications of every assignment of values to wffs other than atoms and their negations (so there is no more information to be used) then no contradiction has emerged, and the argument is indeed invalid.

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- So, for example, to test an inference, we now start by assuming that the premisses are true and the negation of the conclusion is true too.
- Why? Because if every wff on a tree is marked ‘T’, we eventually needn't bother to keep writing down those explicit assignments.

# The unpacking rules – 1

The straight rules are, schematically, as follows:

$$\begin{array}{c}(\phi \land \psi) \Rightarrow \mathsf{T} \\ | \\ \phi \Rightarrow \mathsf{T} \\ \psi \Rightarrow \mathsf{T}\end{array}$$

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$$(\phi \land \psi) \Rightarrow \mathsf{T}$$
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$$\phi \Rightarrow \mathsf{T}$$
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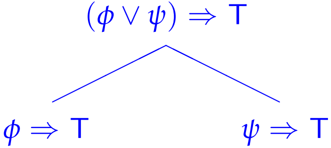
$$\begin{array}{c} \neg\neg\phi \Rightarrow \mathsf{T} \\ | \\ \phi \Rightarrow \mathsf{T} \end{array}$$

# The unpacking rules – 2

The branching rules are, schematically, as follows:



$(\phi \vee \psi) \Rightarrow \mathsf{T}$

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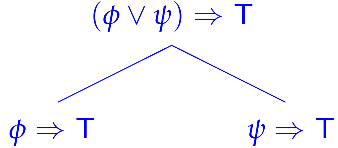

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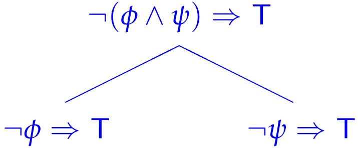

$\neg(\phi \wedge \psi) \Rightarrow \mathsf{T}$

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## Example 1

$$(P \vee Q),\ \neg(Q \wedge \neg R),\ \text{So } (R \vee P)$$

# Example 1

$$(P \lor Q),\ \neg(Q \land \neg R),\ \text{So}\ (R \lor P)$$

$$(P \lor Q) \Rightarrow \mathsf{T}$$
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Heuristic tip: apply straight rules first when we can!

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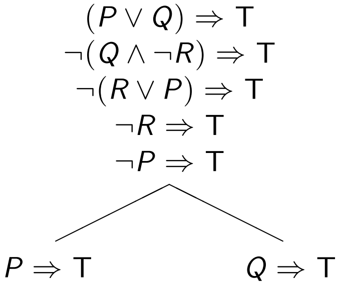

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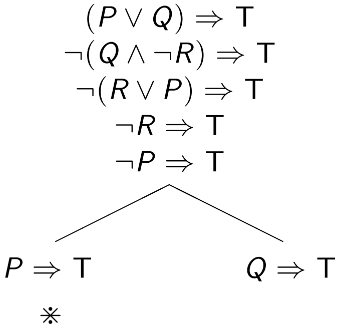

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$\divideontimes$

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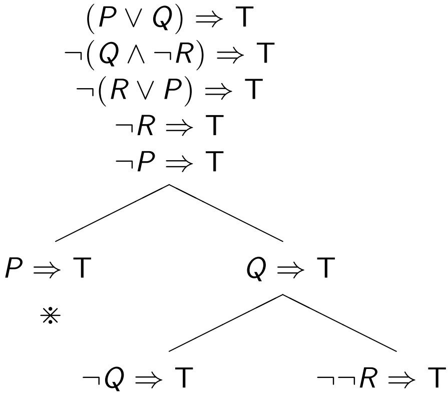

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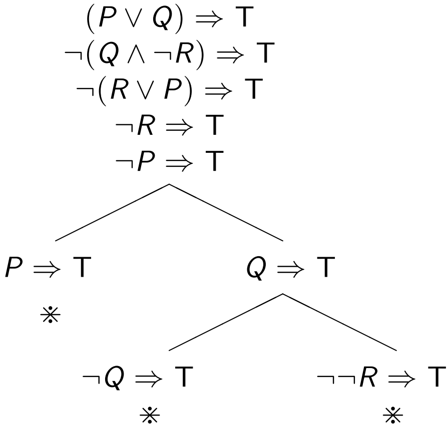

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$\divideontimes$

$\neg Q \Rightarrow \mathsf{T}$ $\quad$ $\neg\neg R \Rightarrow \mathsf{T}$

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# Example 2

$$\neg(P \land Q),\ (R \lor Q),\ \text{So}\ \neg(R \land \neg P)$$

## Example 2

$$\neg(P \wedge Q),\ (R \vee Q),\ \text{So}\ \neg(R \wedge \neg P)$$

$$\neg(P \wedge Q) \Rightarrow \mathsf{T}$$
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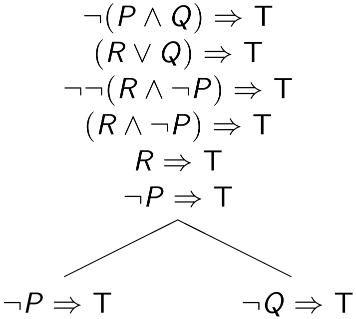

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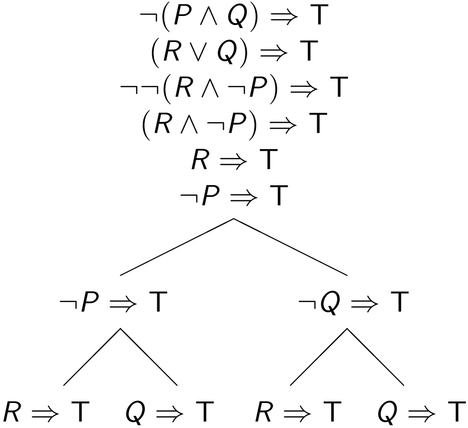

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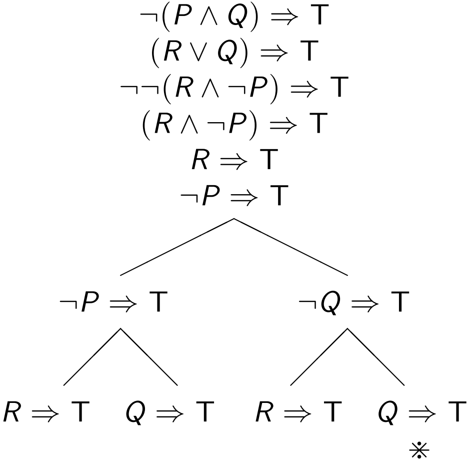

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$\divideontimes$

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$$Q,\ \neg(\neg R \land Q),\ \text{So } R$$

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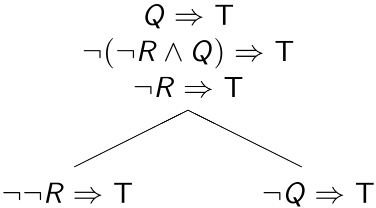

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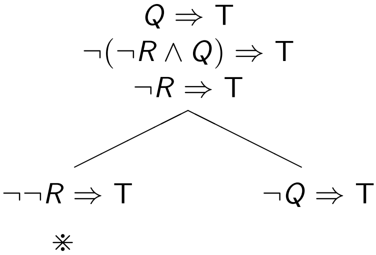

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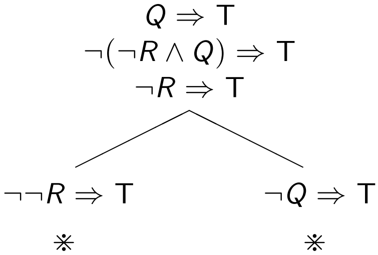

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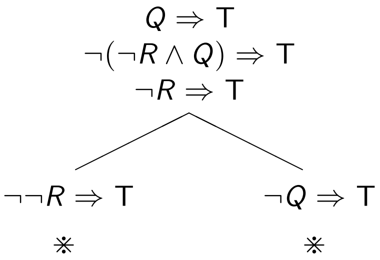

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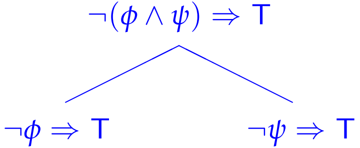
NB: Why does the double negation appear on the left? Because the rule for unpacking negated conjunctions always introduces new negation signs:



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$$\neg(P \vee Q),\ (\neg R \vee P),\ (R \vee S) \text{ So } (S \wedge \neg Q)$$

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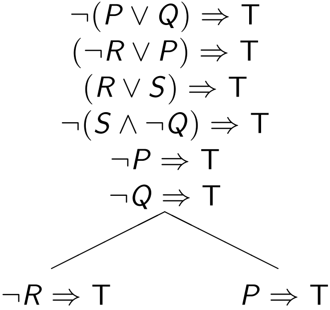

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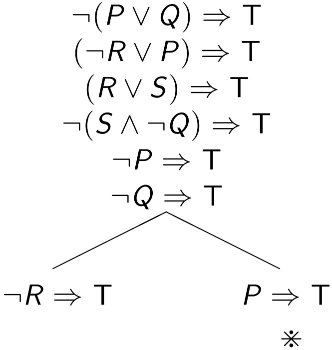

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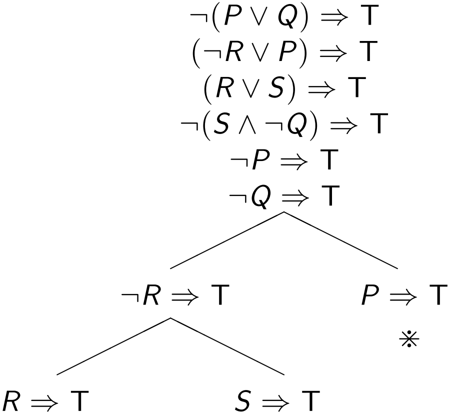

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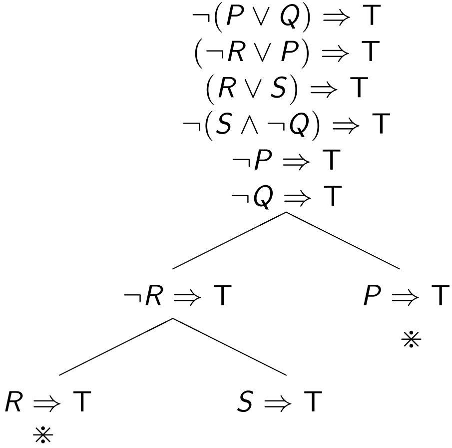

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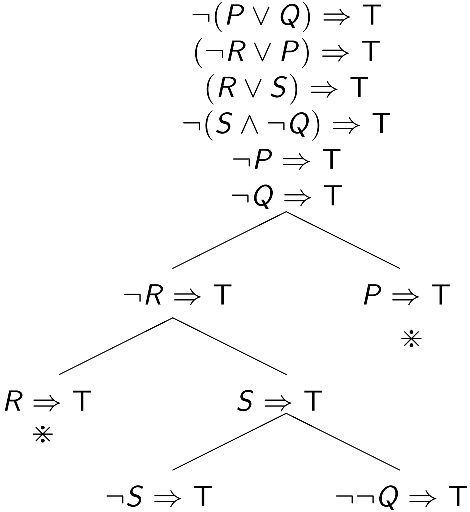

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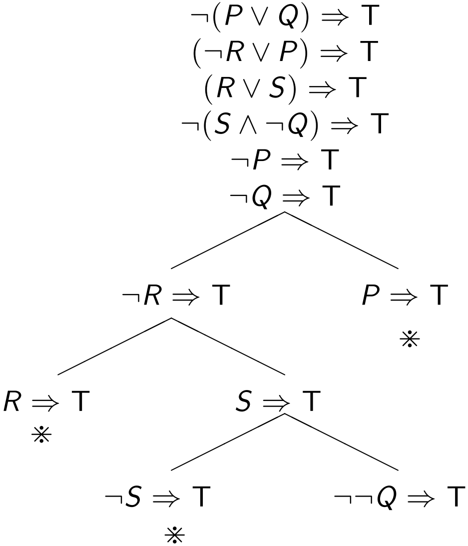

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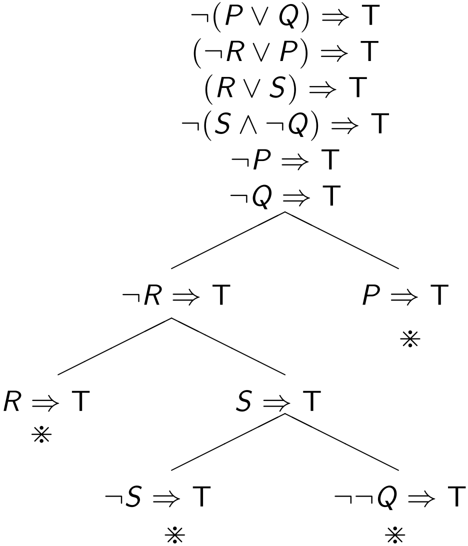

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- Single-signed trees
- Unsigned trees
- Extending trees to cover conditionals
- Finally, the rules for biconditionals

# Dropping the explicit assignments of truth-values

- We introduced trees as a way of setting out a ‘working backwards’ test for tautological validity: assume the premisses are all true and the conclusion false, and see what follows from those assumptions – exploring branching alternatives when necessary.

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- We introduced trees as a way of setting out a ‘working backwards’ test for tautological validity: assume the premisses are all true and the conclusion false, and see what follows from those assumptions – exploring branching alternatives when necessary.
- Our first informal, unofficial trees (naturally enough) had some wffs assigned the value True and others assigned the value False. But in presenting official rules for trees, we’ve managed to assign every wff on a tree the value True – by the simple expedient of always saying ‘$\neg\phi \Rightarrow$ T’ rather than ‘$\phi \Rightarrow$ F’.
- Since on our official signed trees the assignments of truth-value are always assignments of the value True, we can henceforth simply suppress those explicit assignments.

## The unpacking rules again – 1

The straight rules are, schematically, as follows:

$$\begin{array}{c}(\phi \land \psi)\\ | \\ \phi \\ \psi\end{array}$$

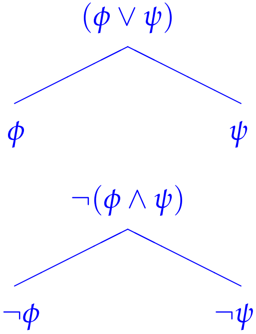
$$\begin{array}{c}\neg(\phi \lor \psi)\\ | \\ \neg\phi \\ \neg\psi\end{array}$$

$$\begin{array}{c}\neg\neg\phi\\ | \\ \phi\end{array}$$

The rules tell you what else must be true, given a complex wff assumed true.

# The unpacking rules again – 2

The branching rules are, schematically, as follows:



$(\phi \vee \psi)$

$\phi$ $\psi$

$\neg(\phi \wedge \psi)$

$\neg\phi$ $\neg\psi$

The rules present alternatives, one of which must be true, give a complex wff assumed true.

## The structural rules

- Start the tree with the premisses and negation of the conclusion.

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- Close a branch with the absurdity marker if it contains some wff $\varphi$ and also its $\neg\varphi$ negation.

If every branch of the tree closes the argument is valid. Otherwise, if a branch stays open even when every complex wff is unpacked, the argument is invalid.

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- It doesn't matter which order you unpack complex wffs in . . .
- But obviously a good tactic is to apply 'straight' rules first when you can, to avoid sprawling trees as far as you can.
- Another good tactic is, when you have the choice, apply branching rule which more quickly closes off a branch.

# A first example

$$P,\ \neg(P \wedge \neg Q),\ (Q \vee R) \text{ So } R$$

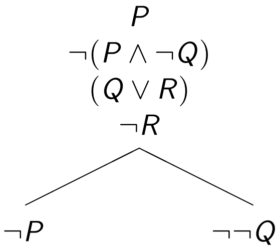
# A first example

$$P,\ \neg(P \wedge \neg Q),\ (Q \vee R) \text{ So } R$$

$$P$$
$$\neg(P \wedge \neg Q)$$
$$(Q \vee R)$$
$$\neg R$$

# A first example

$P$, $\neg(P \wedge \neg Q)$, $(Q \vee R)$ So $R$



$P$

$\neg(P \wedge \neg Q)$

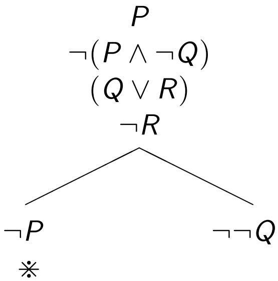
$(Q \vee R)$

$\neg R$

$\neg P$ $\neg\neg Q$

# A first example

$P$, $\neg(P \wedge \neg Q)$, $(Q \vee R)$ So $R$



$P$

$\neg(P \wedge \neg Q)$

$(Q \vee R)$

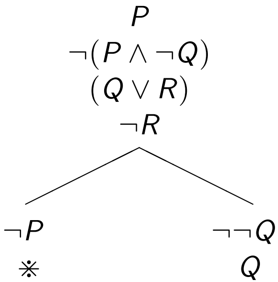
$\neg R$

$\neg P$ ※

$\neg\neg Q$

# A first example

$P$, $\neg(P \wedge \neg Q)$, $(Q \vee R)$ So $R$



$P$

$\neg(P \wedge \neg Q)$

$(Q \vee R)$

$\neg R$

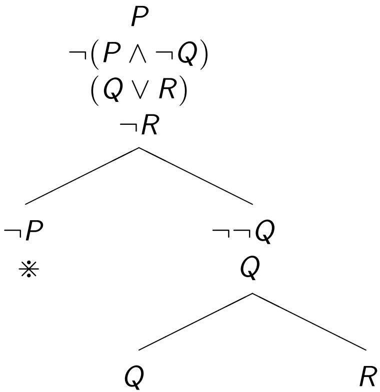
$\neg P$ ※

$\neg\neg Q$

$Q$

# A first example

$P$, $\neg(P \wedge \neg Q)$, $(Q \vee R)$ So $R$



$P$

$\neg(P \wedge \neg Q)$

$(Q \vee R)$

$\neg R$

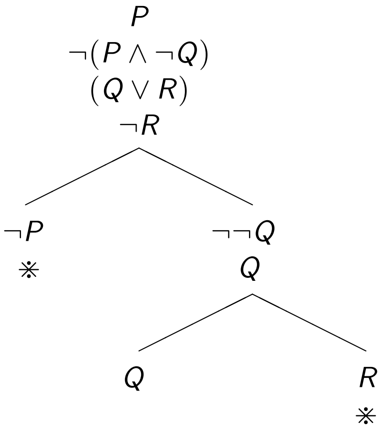
$\neg P$ $\neg\neg Q$

※ $Q$

$Q$ $R$

# A first example

$P$, $\neg(P \wedge \neg Q)$, $(Q \vee R)$ So $R$



$P$

$\neg(P \wedge \neg Q)$

$(Q \vee R)$

$\neg R$

$\neg P$

$\divideontimes$

$\neg\neg Q$

$Q$

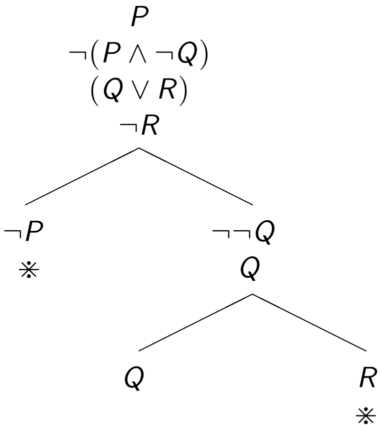
$Q$

$R$

$\divideontimes$

# A first example

$$P,\ \neg(P \wedge \neg Q),\ (Q \vee R) \text{ So } R$$



$P$

$\neg(P \wedge \neg Q)$

$(Q \vee R)$

$\neg R$

$\neg P$ $\qquad$ $\neg\neg Q$

$\divideontimes$ $\qquad$ $Q$

$Q$ $\qquad$ $R$

$\divideontimes$

NB we can read off from the open branch that to make the premisses and negated conclusion true it suffices to make $P$, $Q$ and $\neg R$ all true.

# Another example

$$\neg(P \land \neg Q),\ (\neg Q \lor R)\ \text{So}\ (R \lor \neg P)$$

# Another example

$$\neg(P \land \neg Q),\ (\neg Q \lor R)\ \text{So}\ (R \lor \neg P)$$

$$\neg(P \land \neg Q)$$
$$(\neg Q \lor R)$$
$$\neg(R \lor \neg P)$$

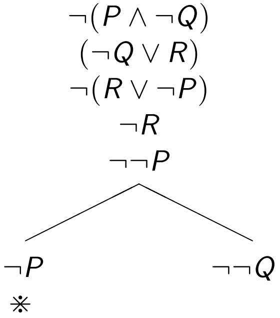
# Another example

$$\neg(P \wedge \neg Q),\ (\neg Q \vee R)\ \text{So}\ (R \vee \neg P)$$

$$\begin{array}{c} \neg(P \wedge \neg Q) \\ (\neg Q \vee R) \\ \neg(R \vee \neg P) \\ \neg R \\ \neg\neg P \end{array}$$

# Another example

$$\neg(P \land \neg Q),\ (\neg Q \lor R) \text{ So } (R \lor \neg P)$$



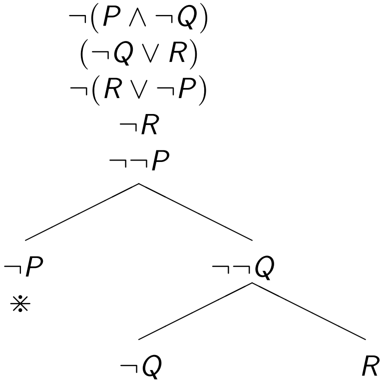
$$\neg(P \land \neg Q)$$
$$(\neg Q \lor R)$$
$$\neg(R \lor \neg P)$$
$$\neg R$$
$$\neg\neg P$$

$\neg P$ ⋇

$\neg\neg Q$

# Another example

$\neg(P \wedge \neg Q),\ (\neg Q \vee R)$ So $(R \vee \neg P)$



$\neg(P \wedge \neg Q)$

$(\neg Q \vee R)$

$\neg(R \vee \neg P)$

$\neg R$

$\neg\neg P$

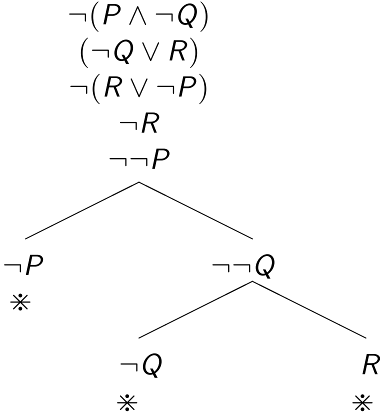
$\neg P$ | $\neg\neg Q$

⋇

$\neg Q$ | $R$

# Another example

$\neg(P \wedge \neg Q)$, $(\neg Q \vee R)$ So $(R \vee \neg P)$



$\neg(P \wedge \neg Q)$

$(\neg Q \vee R)$

$\neg(R \vee \neg P)$

$\neg R$

$\neg\neg P$

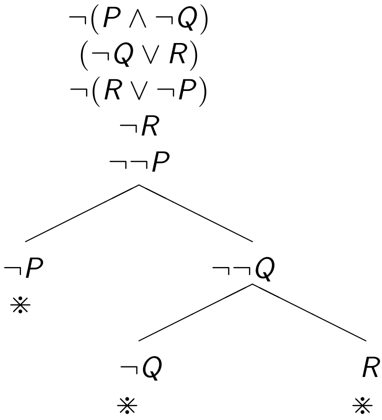
$\neg P$ ※

$\neg\neg Q$

$\neg Q$ ※

$R$ ※

$\neg(P \wedge \neg Q), (\neg Q \vee R)$ So $(R \vee \neg P)$



$\neg(P \wedge \neg Q)$

$(\neg Q \vee R)$

$\neg(R \vee \neg P)$

$\neg R$

$\neg\neg P$

$\neg P$ ※

$\neg\neg Q$

$\neg Q$ ※

$R$ ※

NB: even in this case (with just three atoms in play) you will have finished the tree and decided the inference is valid before you'd even have completed writing out the frame of truth table (the list of possible assignments down the side and the premisses and conclusion along the top)!

# A further example to try

$$\neg(P \lor Q),\ (\neg R \lor P),\ (R \lor S) \text{ So } (S \land \neg Q)$$

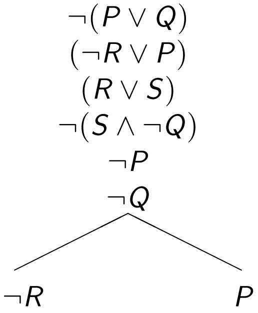
# A further example to try

$$\neg(P \lor Q)$$
$$(\neg R \lor P)$$
$$(R \lor S)$$
$$\neg(S \land \neg Q)$$

# A further example to try

$$\neg(P \lor Q)$$
$$(\neg R \lor P)$$
$$(R \lor S)$$
$$\neg(S \land \neg Q)$$
$$\neg P$$
$$\neg Q$$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

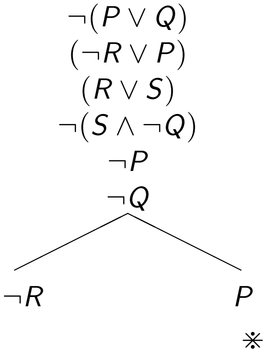
$\neg(S \land \neg Q)$

$\neg P$

$\neg Q$

$\neg R$ $\quad$ $P$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

$\neg(S \land \neg Q)$

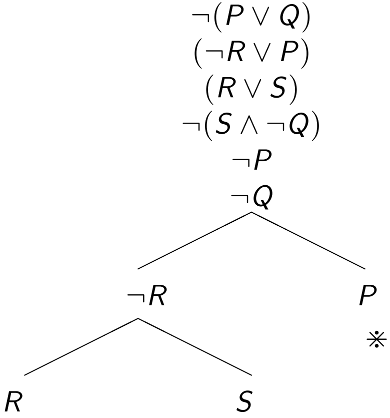
$\neg P$

$\neg Q$

$\neg R$ $\quad$ $P$

$\divideontimes$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

$\neg(S \land \neg Q)$

$\neg P$

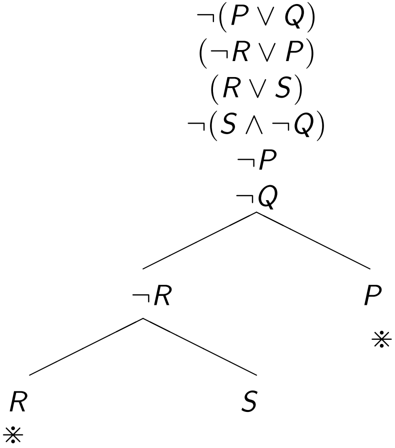
$\neg Q$

$\neg R$ $\quad$ $P$

$\divideontimes$

$R$ $\quad$ $S$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

$\neg(S \land \neg Q)$

$\neg P$

$\neg Q$

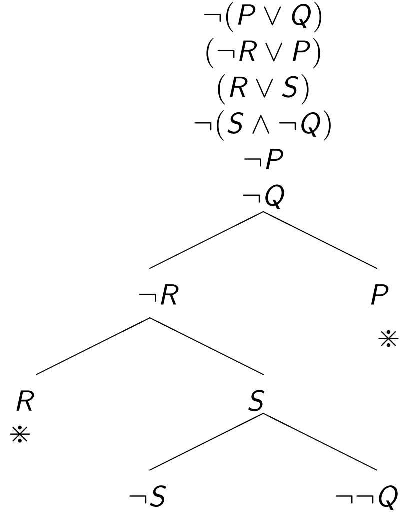
$\neg R$ $P$

⁂

$R$ $S$

⁂

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

$\neg(S \land \neg Q)$

$\neg P$

$\neg Q$

$\neg R$ $\quad$ $P$

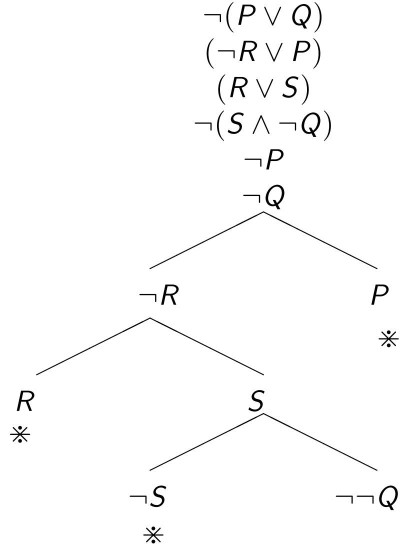
$\divideontimes$

$R$ $\quad$ $S$

$\divideontimes$

$\neg S$ $\quad$ $\neg\neg Q$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

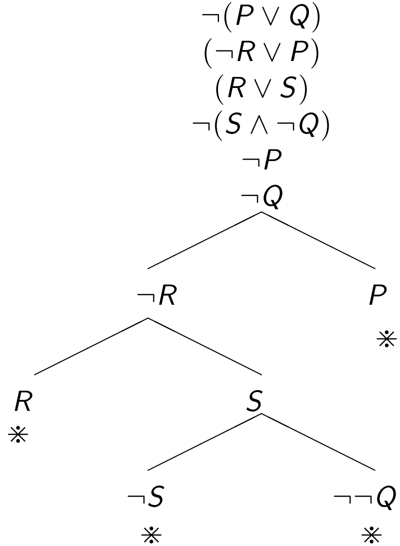
$(R \lor S)$

$\neg(S \land \neg Q)$

$\neg P$

$\neg Q$

# A further example to try



$\neg(P \lor Q)$

$(\neg R \lor P)$

$(R \lor S)$

$\neg(S \land \neg Q)$

$\neg P$

$\neg Q$

$\neg R$ $\quad$ $P$

※

$R$ $\quad$ $S$

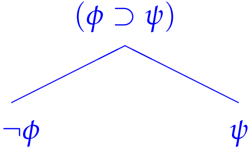
※

$\neg S$ $\quad$ $\neg\neg Q$

※ $\quad$ ※

- Single-signed trees
- Unsigned trees
- Extending trees to cover conditionals
- Finally, the rules for biconditionals

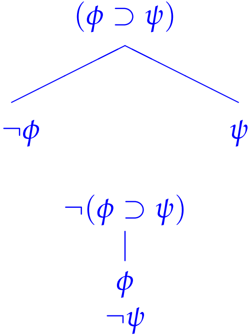
# The unpacking rules for the conditional



$(\phi \supset \psi)$

$\neg\phi$ $\psi$

# The unpacking rules for the conditional



$(\phi \supset \psi)$

$\neg\phi$ $\quad$ $\psi$

$\neg(\phi \supset \psi)$

$\phi$

$\neg\psi$

# A first example of a tree using conditionals

$$(P \supset (Q \supset R)) \text{ So } (Q \supset (P \supset R))$$

## A first example of a tree using conditionals

$$(P \supset (Q \supset R)) \text{ So } (Q \supset (P \supset R))$$

$$(P \supset (Q \supset R))$$
$$\neg(Q \supset (P \supset R))$$

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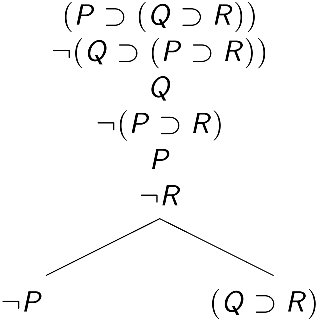
$$(P \supset (Q \supset R)) \text{ So } (Q \supset (P \supset R))$$

$$(P \supset (Q \supset R))$$
$$\neg(Q \supset (P \supset R))$$
$$Q$$
$$\neg(P \supset R)$$

# A first example of a tree using conditionals

$(P \supset (Q \supset R))$ So $(Q \supset (P \supset R))$

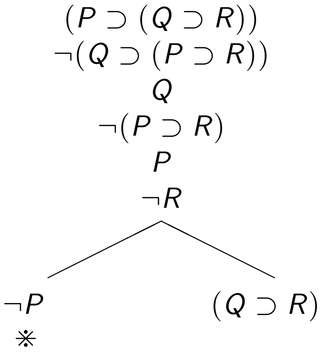


$$(P \supset (Q \supset R))$$
$$\neg(Q \supset (P \supset R))$$
$$Q$$
$$\neg(P \supset R)$$
$$P$$
$$\neg R$$

$\neg P$ $\quad\quad\quad\quad$ $(Q \supset R)$

# A first example of a tree using conditionals

$$(P \supset (Q \supset R)) \text{ So } (Q \supset (P \supset R))$$

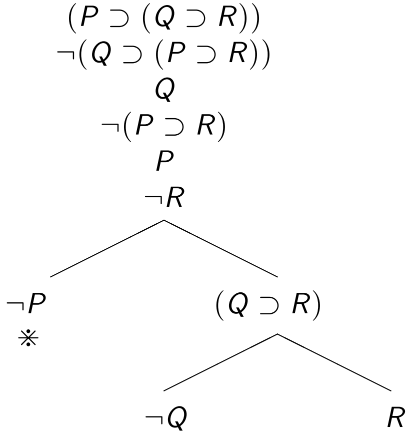


$$(P \supset (Q \supset R))$$
$$\neg(Q \supset (P \supset R))$$
$$Q$$
$$\neg(P \supset R)$$
$$P$$
$$\neg R$$

$\neg P$ ※ | $(Q \supset R)$

# A first example of a tree using conditionals

$(P \supset (Q \supset R))$ So $(Q \supset (P \supset R))$



$(P \supset (Q \supset R))$

$\neg(Q \supset (P \supset R))$

$Q$

$\neg(P \supset R)$

$P$

$\neg R$

$\neg P$ ※

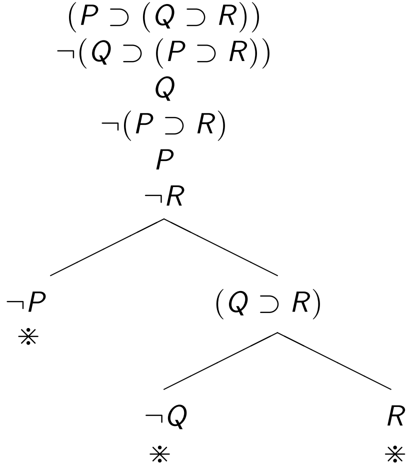
$(Q \supset R)$

$\neg Q$

$R$

# A first example of a tree using conditionals

$$(P \supset (Q \supset R)) \text{ So } (Q \supset (P \supset R))$$



$(P \supset (Q \supset R))$

$\neg(Q \supset (P \supset R))$

$Q$

$\neg(P \supset R)$

$P$

$\neg R$

$\neg P$ $\qquad$ $(Q \supset R)$

$\divideontimes$

$\neg Q$ $\qquad$ $R$

$\divideontimes$ $\qquad$ $\divideontimes$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

$$\begin{gathered}(P \vee Q)\\(P \supset R)\\(Q \supset S)\\\neg(R \vee S)\end{gathered}$$

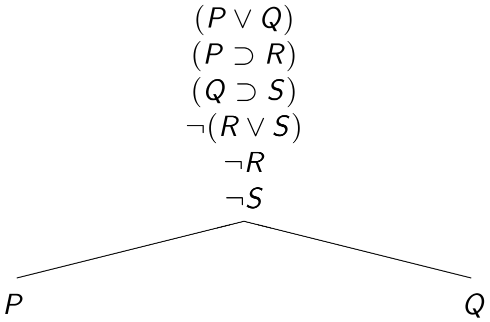
# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

$$\begin{array}{c} (P \vee Q) \\ (P \supset R) \\ (Q \supset S) \\ \neg(R \vee S) \\ \neg R \\ \neg S \end{array}$$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

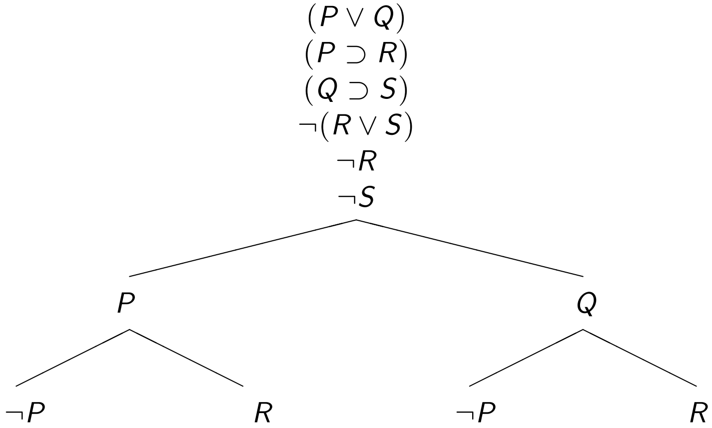
$\neg R$

$\neg S$

$P$ $\quad$ $Q$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

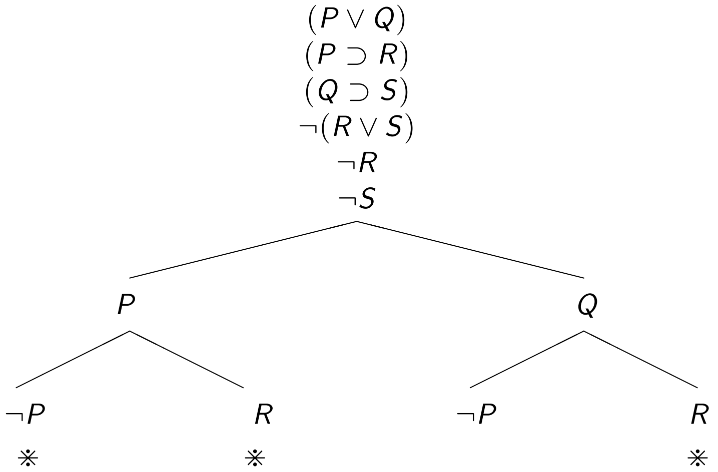
$\neg S$

$P$ $Q$

$\neg P$ $R$ $\neg P$ $R$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

$\neg S$

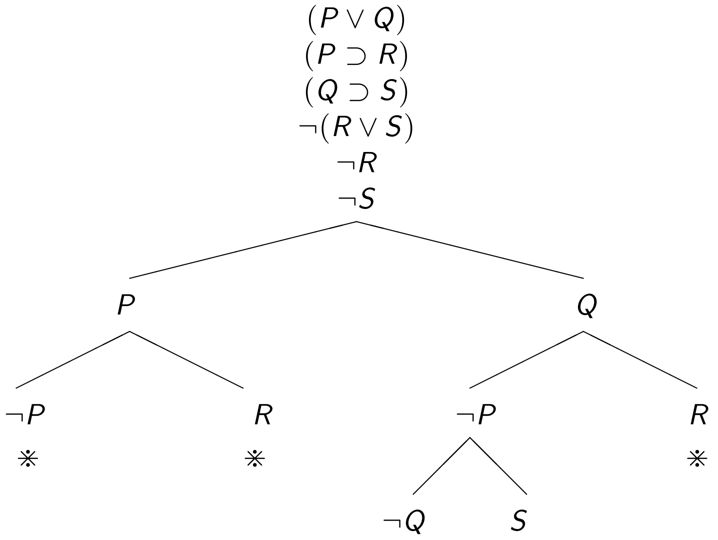
$P$ $Q$

$\neg P$ $R$ $\neg P$ $R$

※ ※ ※

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$
$(P \supset R)$
$(Q \supset S)$
$\neg(R \vee S)$
$\neg R$
$\neg S$

$P$ $Q$

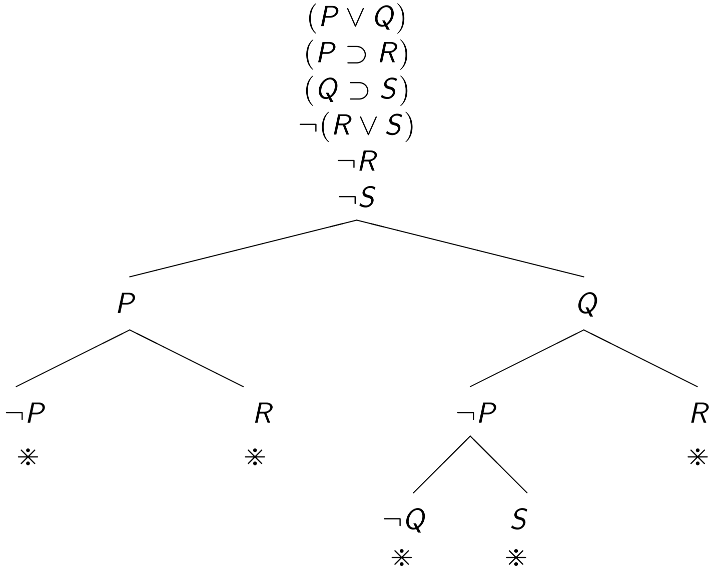
$\neg P$ $R$ $\neg P$ $R$

※ ※ ※

$\neg Q$ $S$

# Another example of a tree using conditionals

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

$\neg S$

$P$ $Q$

$\neg P$ $R$ $\neg P$ $R$

※ ※ ※

$\neg Q$ $S$

※ ※

# Alternative tree for same inference

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

# Alternative tree for same inference

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

$$\begin{array}{c} (P \vee Q) \\ (P \supset R) \\ (Q \supset S) \\ \neg(R \vee S) \end{array}$$

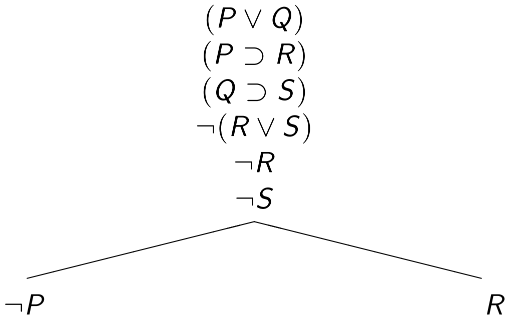
# Alternative tree for same inference

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$

$$\begin{array}{c} (P \vee Q) \\ (P \supset R) \\ (Q \supset S) \\ \neg(R \vee S) \\ \neg R \\ \neg S \end{array}$$

# Alternative tree for same inference

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

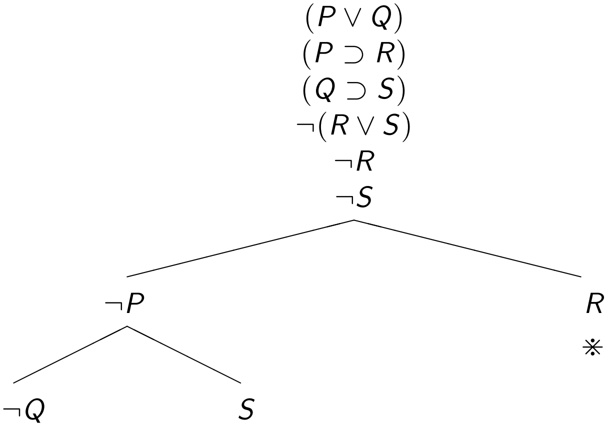
$\neg R$

$\neg S$

$\neg P$ $\quad$ $R$

# Alternative tree for same inference

$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

$\neg S$

$\neg P$ $R$

$\divideontimes$

$\neg Q$ $S$

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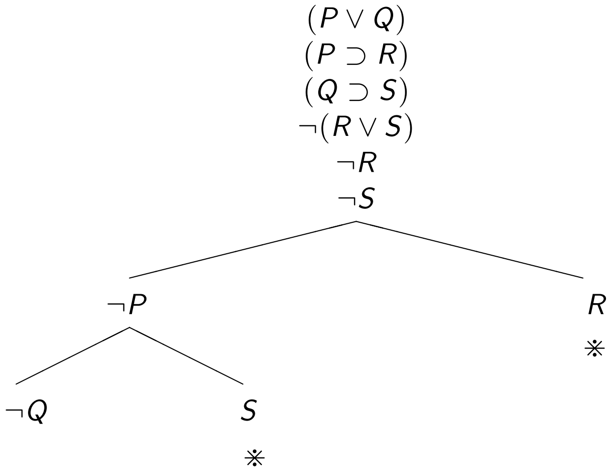
$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

$\neg S$

$\neg P$

$R$

$\divideontimes$

$\neg Q$

$S$

$\divideontimes$

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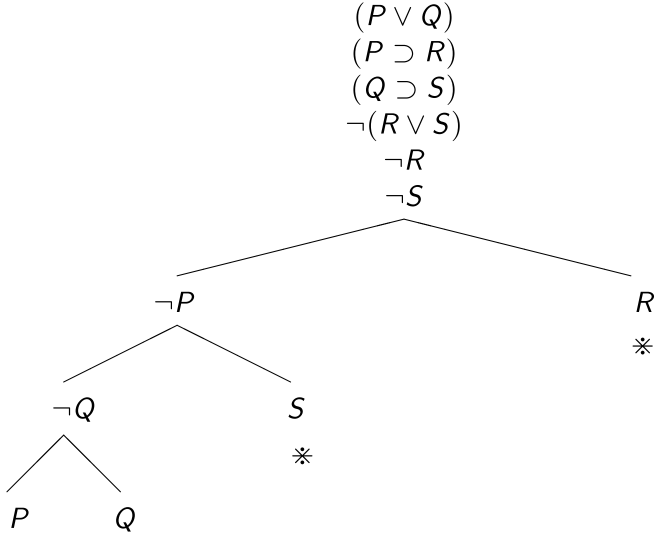

$(P \vee Q)$
$(P \supset R)$
$(Q \supset S)$
$\neg(R \vee S)$
$\neg R$
$\neg S$

$\neg P$
$R$
$\divideontimes$

$\neg Q$
$S$
$\divideontimes$

$P$
$Q$

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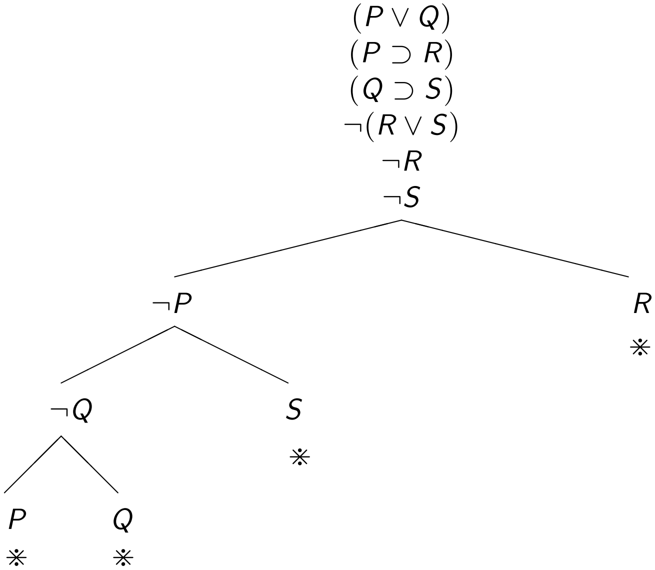
$(P \vee Q)$, $(P \supset R)$, $(Q \supset S)$, So $(R \vee S)$



$(P \vee Q)$

$(P \supset R)$

$(Q \supset S)$

$\neg(R \vee S)$

$\neg R$

$\neg S$

$\neg P$ | $R$ ※

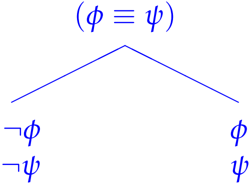
$\neg Q$ | $S$ ※

$P$ ※ | $Q$ ※

# Finally, the rules for biconditionals

- Single-signed trees
- Unsigned trees
- Extending trees to cover conditionals
- Finally, the rules for biconditionals

# The unpacking rules for the biconditional

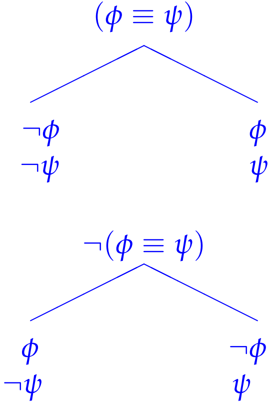


$(\phi \equiv \psi)$

$\neg\phi$ $\phi$

$\neg\psi$ $\psi$

# The unpacking rules for the biconditional



$(\phi \equiv \psi)$

$\neg\phi$ $\quad$ $\phi$

$\neg\psi$ $\quad$ $\psi$

$\neg(\phi \equiv \psi)$

$\phi$ $\quad$ $\neg\phi$

$\neg\psi$ $\quad$ $\psi$

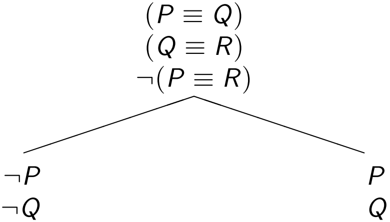
# An example

$$(P \equiv Q),\ (Q \equiv R)\ \text{So}\ (P \equiv R)$$

## An example

$$(P \equiv Q)$$
$$(Q \equiv R)$$
$$\neg(P \equiv R)$$

# An example

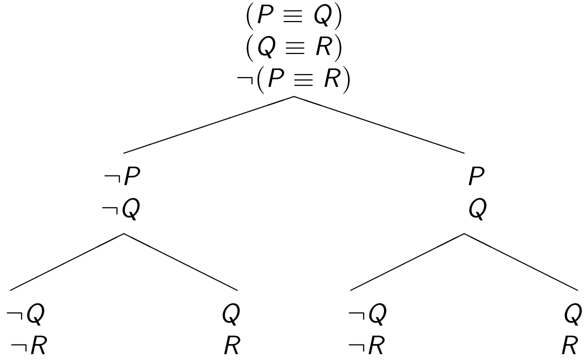


$(P \equiv Q)$
$(Q \equiv R)$
$\neg(P \equiv R)$

$\neg P$
$\neg Q$

$P$
$Q$

# An example



$(P \equiv Q)$

$(Q \equiv R)$

$\neg(P \equiv R)$

$\neg P$

$\neg Q$

$P$

$Q$

$\neg Q$

$\neg R$

$Q$

$R$

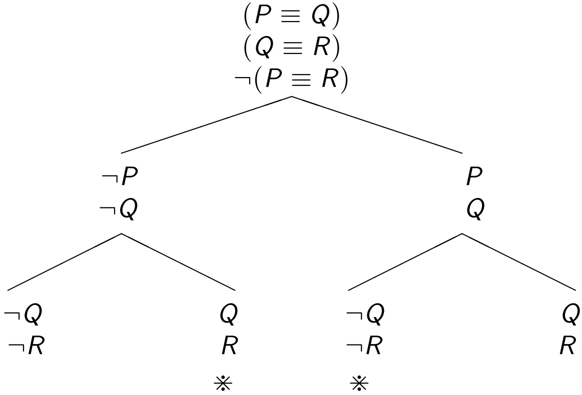
$\neg Q$

$\neg R$

$Q$

$R$

# An example



$(P \equiv Q)$
$(Q \equiv R)$
$\neg(P \equiv R)$

$\neg P$
$\neg Q$

$P$
$Q$

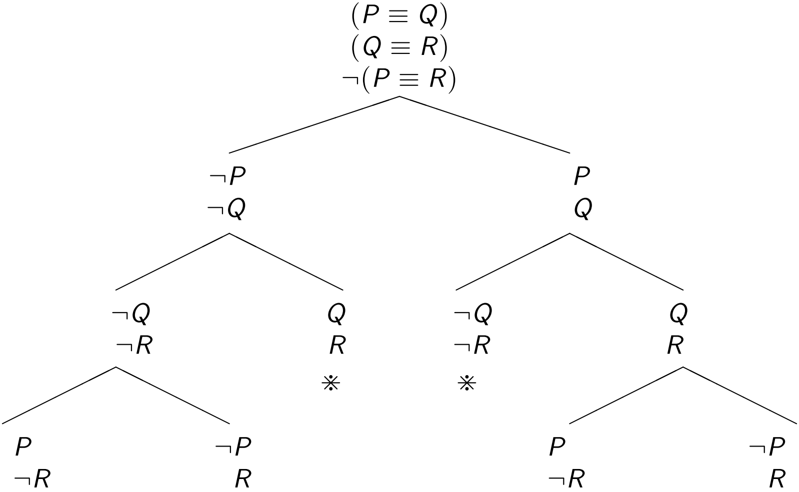
$\neg Q$
$\neg R$

$Q$
$R$
$\divideontimes$

$\neg Q$
$\neg R$
$\divideontimes$

$Q$
$R$

# An example



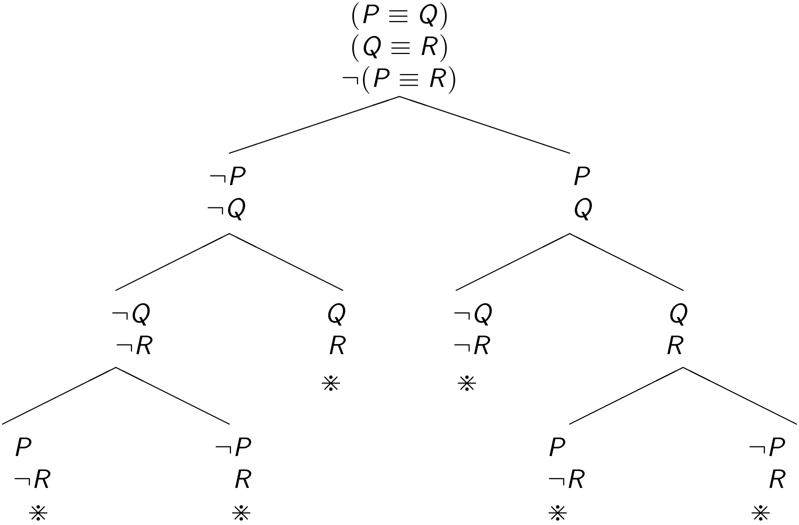
$(P \equiv Q)$
$(Q \equiv R)$
$\neg(P \equiv R)$

$\neg P$ $\neg Q$ | $P$ $Q$

$\neg Q$ $\neg R$ | $Q$ $R$ $\divideontimes$ | $\neg Q$ $\neg R$ $\divideontimes$ | $Q$ $R$

$P$ $\neg R$ | $\neg P$ $R$ | $P$ $\neg R$ | $\neg P$ $R$

# An example



$(P \equiv Q)$
$(Q \equiv R)$
$\neg(P \equiv R)$

$\neg P$ $\quad$ $P$
$\neg Q$ $\quad$ $Q$

$\neg Q$ $\quad$ $Q$ $\quad$ $\neg Q$ $\quad$ $Q$
$\neg R$ $\quad$ $R$ $\quad$ $\neg R$ $\quad$ $R$
$\divideontimes$ $\quad$ $\divideontimes$

$P$ $\quad$ $\neg P$ $\quad$ $P$ $\quad$ $\neg P$
$\neg R$ $\quad$ $R$ $\quad$ $\neg R$ $\quad$ $R$
$\divideontimes$ $\quad$ $\divideontimes$ $\quad$ $\divideontimes$ $\quad$ $\divideontimes$

## Two comments

- We can use trees to deal with arguments involving other truth-functional connectives too, like the exclusive disjunction (which we haven't added to our official language). What the rules for that would can be left as a mini-exercise to explore.

# Two comments

- We can use trees to deal with arguments involving other truth-functional connectives too, like the exclusive disjunction (which we haven't added to our official language). What the rules for that would can be left as a mini-exercise to explore.
- But for now two general comments on trees, one related to the P vs NP problem, one related to so-called 'natural deduction'.