32.4 Empty domains?

We finish this chapter by turning from illustrations of strategies for proof-construction to consider a basic issue of principle, one which we have so far passed quietly by.

(a) Start from an example. Tachyons are, by definition, elementary particles which are superluminal, i.e. which travel faster than the speed of light. So, adopting a QL language quantifying over elementary particles, and with the obvious predicates, the following is true:

(1) $\forall x(Tx \to Sx)$.

Now, given that truth, we *can't* use our QL inference rules to deduce

(2) $\exists x(Tx \land Sx)$.

Which is just as it should be: see §29.2. And also note that (2) *does* trivially entail

(3) $\exists x\, Sx$.

And we don't want the truth (1) – *if* something is a tachyon, it is superluminal – to entail a proposition that tells us that there actually *are* some superluminal particles.

So far so good. But what happens if, in the spirit of Quine's maxim of shallow analysis, we think along the following lines? – the current topic is just tachyons, so we can keep things simple by taking our domain to be just them (compare our handling of argument **F**). So, re-interpreting the quantifiers to run over just that domain, the claim that tachyons are superluminal can be regimented more simply as

(1′) $\forall x\, Sx$.

Hold on though! We *do* have the QL mini-proof

(1′)	$\forall x\, Sx$	(Prem)
(2′)	Sa	($\forall$E 1)
(3′)	$\exists x\, Sx$	($\exists$I 2)

And lo and behold, we now seem to have proved the existence of something that travels faster than light from a reformulation of what was originally supposed to be a premiss that is true by definition. What's gone wrong?

(b) There's no mystery here! Look at the proof. What happens at the second step? We take an arbitrary member of the current domain and temporarily dub it 'a'. But we can only do *that* if the domain is populated and contains some object(s) – at least one – to choose from. In other words, *our logical apparatus, and in particular the way it handles dummy names, presumes that the universe of discourse isn't empty.*

Therefore, in (i) taking the domain to comprise tachyons, and also (ii) assuming that we can use our current QL rules, we are implicitly already assuming that there *are* some tachyons, and hence some superluminal particles. But, as far as we know, there are no such things.

When we first discussed interpreting the QL quantifier-variable notation in §27.3, we said that we will fix what the quantifiers of a language range over by giving a description *D* of the domain. But we added, without explanation at the time, that we aren't allowed to give just *any* domain description *D* (like 'tachyons'): there must be *something* satisfying the description. In short, we ban empty domains. Now we know why. If we allow empty domains, our – entirely standard – QL inference rules won't be correct.

Look at it this way:

(1) In an empty domain, '∀x Sx' is always true, whatever condition *C* the predicate expresses – because it is always true that, if *n* is in the domain, *n* is *C* (because the antecedent of that conditional is always false).

(2) In an empty domain, '∃x Sx' is always false.

(3) But as we have just seen, our quantifier rules give us a proof from '∀x Sx' to '∃x Sx' – which, in an empty domain, takes us from truth to falsehood.

(c) This much, then, is uncontroversial: when we apply our current quantifier rules to construct derivations in some QL language, we are implicitly assuming that its universe of discourse is not empty. But how should we respond to this fact? Now the controversy starts! Consider this line of argument:

> An inference is *logically* valid if it is necessarily truth-preserving in virtue of topic-neutral features of its structure (see §6.2). And *formal* logic is the study of logical validity, using regimented languages to enable us to bring out how arguments of certain forms are valid irrespective of their subject-matter. So our formal logic of quantificational reasoning should aim to provide principles of reasoning that are valid whether we are talking about e.g. electrons (that do exist) or e.g. tachyons (that don't). That is to say, one thing our formal logic should be neutral about is whether our current domain of quantification is populated or not. After all, we can want to argue logically about things that we believe don't exist, precisely in order to try to show that they don't exist! While other times we can want to argue in an exploratory way, in ignorance of whether what we are talking about exists. Other times again, we argue about the properties of things we are already convinced exists. And we presumably want to formally regiment correct forms of inference which we can apply neutrally across these different cases. Hence *one* way our formal logic should be topic neutral is by allowing empty domains. Hence our current QL rules – being incorrect for empty domains – are not topic neutral, don't correctly capture only *logical* validities, and so need revision.

However, the notion of topic-neutrality isn't entirely clear (see §6.4). And it isn't clear either how far logic *should* be topic neutral (we wondered about this when we mentioned intuitionism in §24.7(d)). So consider this reply:

> Yes, we do argue about inhabitants of other planets, tachyons, superstrings, and other doubtful entities. But such reasoning typically proceeds – or can be regarded as proceeding – in a suppositional mode. For example, we might bracket our set-theoretic investigations with an unspoken 'Let's take it, for the sake of argument, that there *is* this wildly infinitary universe that the mathematician's standard set theory talks about . . . ' (we don't worry too much about the metaphysical status of set theory while actually doing the maths!). Or we be can quite explicitly agnostic: 'Who knows whether there are superstrings? The empirical evidence is minimal. Still, here's a quite lovely theory postulating such things . . . '. Or we can be out-

> and-proud unbelievers, as when we reason about an avowedly fictional world: 'Let's pretend . . . '. Still, *within the scope of the bracketing supposition*, we argue as if we are dealing with suitably populated worlds. Yes, we want the formal principles of reasoning we use to be neutral about the particular topic; but we don't require them to be neutral about whether – even within the scope of the current supposition – we *have* a topic! So a topic-neutral logic need not countenance empty domains.

Still, it will be said, this sort of response misses the central worry about adopting quantifier rules that presuppose that domains are populated:

> Take the following proof using the current QL rules of inference.
>
> | | | |
> |---|---|---|
> | (1) | | |
> | (2) | $(Fa \lor \neg Fa)$ | (LEM) |
> | (3) | $\exists x(Fx \lor \neg Fx)$ | ($\exists$I 2) |
>
> Whatever language we are working in, whatever the domain, we can prove instances of excluded middle like (2) from no premisses, and so (3) is a *logical theorem*. But (3) requires for its truth that there exists at least one thing in the domain of quantification. And, even when we are working within the supposition that there are superstrings, or whatever, we won't want it to be a *logical truth* that there such things. The standard QL rules deliver theorems which aren't logical truths. *That* is what is unacceptable.

But again, the defender of the standard QL rules has a response:

> Repeatedly, we have seen that our choice of a formal logic involves trade-offs between between costs and benefits. For example, we adopt a simple formal notion of entailment like tautological entailment at the cost of allowing apparently irrelevant inferences to count as valid – in particular, we get explosion. We adopt a simply-behaved conditional apt for *some* logical purposes at the cost of ignoring key features of the use of ordinary-language conditionals. And so it goes.
>
> It is the same here. Our QL theorems can reflect (i) outright necessary truths or, as in our example, just reflect (ii) the fundamental presumption that we aren't arguing about *nothing*. So, inside the system, we lose the distinction between (i) and (ii), a distinction which sometimes we will care about a lot. That's a cost. But the resulting benefit is we get a particularly simple and natural set of quantifier rules which are truth-preserving so long as we are not talking about nothing.

Or so the story goes. The proponent of a *free logic* (a logic free of existence assumptions, so one that allows empty domains and usually also allows 'empty names' that don't have a reference) will in turn protest that developing his preferred sort of logic isn't *that* complicated, and the benefits – including ensuring that all theorems really are logical truths – are worth the small extra effort.

But there we must leave the debate. Once more, we remind ourselves that there is no One True Logic written on tablets of stone. And, having flagged up that we *are* at a choice point, we will again continue down the conventional path, as befits an introductory text. So we will stick with the QL rules as we have presented them. Our quantificational logic, then, is the logic of reasoning about non-empty domains.