Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 6: Logical validity

Which of the following arguments are deductively valid? Which are logically valid? (Defend your answers, as best you can.)

(1) *Only logicians are wise. Some philosophers are not logicians. All who love Aristotle are wise. Hence some of those who don't love Aristotle are still philosophers.*

We have the simple informal derivation:

(1)	Only logicians are wise.	(premiss)
(2)	Some philosophers are not logicians.	(premiss)
(3)	All who love Aristotle are wise.	(premiss)
(4)	All who love Aristotle are logicians.	(from 1, 3)
(5)	Some philosophers do not love Aristotle.	(from 2, 5)
(6)	Some of those who don't love Aristotle are philosophers.	(from 6)

The only inferential principles we are appealing to here are general logical ones concerning the topic-neutral 'only', 'some', 'all', 'not'. So this inference is not just deductively valid but logically valid.

(2) *The Battle of Hastings happened before the Battle of Waterloo. The Battle of Marathon happened before the Battle of Hastings. Hence the Battle of Marathon happened before the Battle of Waterloo.*

Evidently deductively valid – if the premisses are true, then the conclusion has to be true too, just in virtue of the meaning of 'happened before'.

But 'happened before', although of wide application, is not topic-neutral (it can only apply to events in time); so this inference won't count as logically valid.

(3) *Jane is no taller than Jill, Jill is no taller than Jo, Jo is no taller than Jane. So Jane, Jill and Jo are the same height.*

The first two premisses entail that Jane is no taller than Jo (that's not a purely logical entailment, as it depends on the particular meaning of the non-topic-neutral 'no taller than'). And this interim conclusion that Jane is no taller than Jo combined with the third premiss entails that Jane and Jo are the same height (again that that's not a purely logical entailment, as it depends on the particular meanings of the non-topic-neutral 'no taller than' and 'is the same height as').

So we can deduce that Jane and Jo are the same height; and similarly for the other pairs. Hence the premisses deductively entail the conclusion, though they don't *logically* entail it (they don't entail it in virtue of the distribution of top-neutral logical operators in the premisses and conclusion).

(4) *Jane is taller than Jill, Jill is taller than Jo, Jo is taller than Jane. So Jane, Jill and Jo are the same height.*

The first two premisses entail that Jane is taller than Jo. But that is inconsistent with the third premiss. There is no possible way the premisses can be true together. (That's a result of the meaning of the non-topic-neutral 'is taller than').

Since there is no possible way for the premisses to be true together, there is no possible way for the premisses to be true together and conclusion false. Given the classical definition of validity, this argument counts as *valid* (but not logically valid).

(5) *Someone loves Alex, but Alex loves no-one. The person who loves Dr Jones, if anyone does, is Dr Jones. So Alex isn't Dr Jones.*

Consider the following line of argument. We make a temporary supposition at line (4) which leads to absurdity, hence has to be rejected by a reductio argument.

(1)	Someone loves Alex.	(premiss)
(2)	Alex loves no-one.	(premiss)
(3)	The person who loves Dr Jones, if anyone does, is Dr Jones.	(premiss)
(4)	Alex is Dr Jones.	(temporary supposition)
(5)	Someone loves Dr Jones	(from 1, 4)
(6)	Dr Jones loves Dr Jones	(from 3, 5)
(7)	Alex loves Dr Jones	(from 4, 6)
(8)	Contradiction!	(from 2, 7)
(9)	Alex isn't Dr Jones	(from 4–8, by RAA)

This shows that our inference is deductively valid. But is it logically valid? Well apart from moves that depend on the logic of 'someone' and 'anyone', the only other principles we use are when we infer from an identity claim of the form a is j that what is true of a is true of j (line 5), or that what is true of j is true of a (line 7). So what is involved here is the identity relation expressed by 'is' (meaning *is one and the same as*, or *is none other than*). This has the kind of generality and topic-neutrality that means that the 'is' of identity is usually counted as a bit of logical apparatus. So this sort of argument is standardly counted as *logically* valid.

(6) *Whoever respects Socrates respects Plato too. All who respect Euclid respect Aristotle. No one who respects Plato respects Aristotle. Therefore Jo respects Euclid if she doesn't respect Socrates.*

Short answer: it's invalid.

Long answer: Suppose we set out to try to derive the conclusion. We can start like this, chaining together two bits of syllogistic reasoning:

(1)	Whoever respects Socrates respects Plato too.	(premiss)
(2)	All who respect Euclid respect Aristotle.	(premiss)
(3)	No one who respects Plato respects Aristotle.	(premiss)
(4)	No one who respects Socrates respects Aristotle.	(from 1, 3)
(5)	No one who respects Socrates respects Euclid.	(from 2, 4)

This gives us a result relating respecting Euclid and respecting Socrates, which applies to Jo in particular, so we have

(6) If Jo respects Socrates, she doesn't respect Euclid. (from 5)

However the target conclusion has the 'not' in a different place! It says

($6'$) If Jo doesn't respect Socrates, she respects Euclid,

and ($6'$) doesn't follow from (6) – more generally, propositions of the type *if P then not-Q* and *if not-P then Q* are quite different, with neither implying the other.

On the face of it there is no other way of reaching ($6'$), so it looks as if the original argument can't be valid. It's therefore not going to be a waste of time to cast around for a counterexample to the reliability of the relevant pattern of reasoning:

All F are G
All H are J
No G are J
So: if n is not F, n is H.

Here's one, with true premisses and a false conclusion (making sensible assumptions about Jo!):

All logicians are rational people
All grown elephants weigh over a ton
No rational people weigh over a ton
So: if Jo is not a logician, Jo is grown elephant.

(7) *Jill is a good logician only if she admires either Gödel or Gentzen. Jill admires Gödel only if she understands his incompleteness theorem. Whoever admires Gentzen must understand his proof of the consistency of arithmetic. No one can understand Gentzen's proof of the consistency of arithmetic without also understanding Gödel's incompleteness theorem. So if Jill is a good logician, then she understands Gödel's incompleteness theorem.*

Suppose for the sake of argument that Jill *is* a good logician. Then we can argue like this:

> By the first premiss, our supposition implies that either Jill admires Gödel or she admires Gentzen.
>
> So we have two cases to consider. In the first case, Jill admires Gödel so by the second premiss, Jill understands Gödel's incompleteness theorem.
>
> In the second case, Jill admires Gentzen so by the third premiss she must understand Gentzen's proof of the consistency of arithmetic, which by the fourth premiss means that she also understands Gödel's incompleteness theorem.
>
> So in both cases, either way, Jill does understand Gödel's incompleteness theorem.

In sum, our supposition that Jill *is* a good logician leads to the conclusion that she *does* understand Gödel's incompleteness theorem.

Hence, as claimed, if Jill is a good logician then she understands Gödel's incompleteness theorem.

So the given argument is valid, and indeed logically valid, since we have only used principles governing topic-neutral logical notions like 'if' and 'only if', 'whoever', 'no one'

The novelty in this example is that, in defending our answer, we have used a new type of indirect proof which we haven't met before. We want to establish a conditional proposition of the form *if P then Q*. And how have we done this? We have supposed that P is true for the sake of argument. And then shown on the basis of this temporary assumption we can infer Q. This is enough to establish that if indeed P is true, then Q is true too. More about this kind of indirect proof later in the book.

(8) *All the Brontë sisters supported one another. The Brontë sisters were Anne, Charlotte and Emily. Hence Anne, Charlotte and Emily supported one another.*

A valid argument, and surely a *logically* valid argument – for the inference here just relies on the behaviour of 'all' and the 'were' (the past plural of the 'is' of identity).

The interest in this argument is that the standard logical theory for dealing with 'all' that we will meet later in this book can't cope with this sort of argument, because this argument essentially involves terms with plural reference like 'the Brontë sisters' and 'Anne, Charlotte and Emily'. Standard logic, however, only gives a treatment of terms with singular reference, picking out individual items, like 'the parsonage at Haworth' or 'Anne Brontë'.

For note that, in our example, the term 'Anne, Charlotte and Emily' is *not* being used as shorthand, a way of compressing three different claims, one about Anne, one about Charlotte, and one about Emily. Perhaps we can treat 'Anne, Charlotte and Emily lived at Haworth' that way, as a compressed version of 'Anne lived at Haworth and Charlotte lived at Haworth and Emily lived at Haworth.' But 'Anne, Charlotte and Emily supported one another' is obviously not equivalent to the nonsensical 'Anne supported one another and Charlotte supported one another and Emily supported one another '!

(9) *There are exactly two logicians at the party. There is just one literary theorist at the party. No logician is a literary theorist. Therefore, of the party-goers, there are exactly three who are either logicians or literary theorists.*

Evidently valid.

You might be inclined to say, however, that it isn't logically valid, as it depends on the non-logical, arithmetical, truth that one plus two is three.

However, to say that there is exactly one literary theorist at the party is to say: there is a literary theorist at the party and any literary theorist at the party is none other that person. And to say that there are two logicians at the party is to say that there is a logician at the party and there is a distinct logician at the party and any logician there is none other than the former or the latter.

In other words, the so-called numeral quantifiers 'there is exactly one', 'there are exactly two', and likewise 'there are exactly three', can be defined in terms of basic logical notions like 'there is', 'any', 'and', 'or', together with 'is none other than', and its opposite 'is distinct from'.

And later we will show that standard logical principles governing these basic logical notions can be used to derive the conclusion of our argument from its premisses. In other words, we can massage this argument into a form where it is *logically* valid.

(10) *There are no unicorns. Hence the set of unicorns is the empty set.*

If the premiss is true, the conclusion has to be true too! – if there are no unicorns, the set of unicorns won't have any members.

What is not obvious is whether this inference counts as *logically* valid. Does the notion of *set* belong to purview of *logic*? Or is it a substantively *mathematical* notion on a par with (say) *real number* or *group*?

In modern mathematics, set theory is just a theory like any other, dealing with a hierarchical structure of special entities (sets, sets-of-sets, sets-of-sets-of-sets, and so ad infinitum). But key figures in the early development of modern logic such as Frege and Russell did treat the notion of set or class as a specifically logical notion. For example, for Russell initially, talk of the class of Xs was just a way of talking of the Xs. Russell was aware that our ordinary way of talking about the class of Xs can make it look as if we are talking about something over and above the Xs themselves; but he took this to be misleading. So class talk for Russell is just topic-neutral apparatus for talking about many things at once, not apparatus for introducing a special new topic, *sets* in the modern mathematician's sense. (Well, as always with early Russell, things are more complicated than that, but let that pass!)

Obviously we can't pursue the issues here any further. We are just flagging up the unclear status that set talk can have. In a particular context, we will want to ask: is the talk of a set here just a helpful *façon de parler*, a way of talking of many things at once? or does it commit us to a some new special entity with distinctive properties?

(11) *There is water in the cup. Hence there is liquid H_2O in the cup.*

Plainly this isn't a *logically* valid inference. However, there is a widely held metaphysical view that what makes any stuff *water* – the very same stuff as that which falls from the sky as rain, flows in rivers, etc. – is its constitution, which as a matter of fact (we had to discover this!) means being an assembly of H_2O molecules in liquid form. So necessarily, if a cup contains water, then it contains liquid H_2O. Which would make the inference necessarily truth-preserving.

It does, though, sound odd to say that the inference is deductively valid (even if it is so by our definition). To avoid this upshot, we'd have to refine our broad definition of deductively validity – perhaps along the lines of saying that an inference is deductively valid if it is truth-preserving as a matter of conceptual necessity, and then spell out the required notion of conceptual necessity in a way that distinguishes it from a notion of the sort of metaphysical necessity involved in water's being H_2O. As you can imagine, this would be a delicate matter – and certainly something we don't want to be tangling with in an introductory formal logic book. So it is a relief that we will be able to just sidestep this kind of issue by focussing on logical validity. Indeed (as we will soon be discovering), we can focus on just two technical notions which give tidy 'rational reconstructions' of the notion of logical validity for two restricted classes of argument.

(12) *Necessarily, water is* H_2O. *Hence it is not possible that water isn't* H_2O.

As we just noted the premiss here is disputable. But the inference is deductively valid. Given it is necessary that P, then it is not possible that not-P. (And this principle applies whatever species of necessity/possibility is in question, so long as it the same species in the premiss and conclusion. For example, if it is legally necessary for a valid will to be witnessed, then it is not legally possible for a valid will not to be witnessed.)

Is this inference *logically* valid? Well, it is usually supposed that at least the unqualified notion of necessity has the kind of generality and topic-neutrality that makes it a proper object of logical enquiry – and indeed, there is a whole branch of logic, *modal logic*, which is devoted to the logic of notions like necessity and possibility.

(13) *It is possible that it is cold. It is possible that it is rainy. Hence it is possible that it is cold and rainy.*

On the face of it, this inference – another modal inference, involving one of the twin notions of necessity and possibility – is relying on the following schematic pattern of argument:

> It is possible that P. It is possible that Q. Hence it is possible that P *and* Q.

But on a moment's reflection, this is not a correct principle of inference. For example, the following has true premisses and a false conclusion:

> It is possible that Jo has written a book. It is possible that Jo has not written a book. Hence it is possible that Jo both has and has not written a book.

We can conclude, then, that our inference doesn't appeal to a correct modal mode of inference.*

(14) *This argument is valid. Hence this argument is invalid.*

Suppose argument (14) were valid. Then it would have a true premiss and a false conclusion – which makes it invalid. Hence,

(+) Necessarily, if (14) is valid, it follows that it is invalid.

Hence (by the principle that if P implies not-P, it follows that not-P) (14) is invalid. But hold on! we've proved (+), which shows that (14) is valid. So (14) is valid too! Contradiction!

*There is a wrinkle here (skip this on a first reading!) Our official characterization of validity says that an argument is valid if there is no way for the premisses to be true and the conclusion false. Hence if the conclusion of an inference is necessarily true, then – according to this classical definition – the inference will have to count as valid. So consider again the claim that

(×) It is possible that it is cold and rainy,

where we mean 'possible' in the unqualified broadest sense. Can (×) possibly be false? How could it possibly be *impossible* – not even logically possible in the weakest sense, not coherently conceivable – that it is cold and rainy?

More generally, the following modal principle looks defensible, at least for the weak notion of logical possibility.

> If it is possible that P, it is necessarily possible that P.

If this principle *is* acceptable, then (since it is in fact possible that it is cold and rainy), the conclusion (×) of our argument is indeed itself necessary, i.e. cannot be false. So the argument cannot have true premisses and a false conclusion. So by our definition, the argument would count as valid after all. Which is an unwanted conclusion.

We said that we can buy the standard definition of validity despite the fact that it allows an impossibility to validly imply anything, and allows a necessary truth to be validly implied by anything, because that was (usually) a minor defect we could (usually) live with because it buys as neat simplicity elsewhere. Unsurprisingly, however, once we turn to the special case of arguing about impossibility and necessity this minor defect arguably becomes more of a troublesome bug.

This sort of paradox was known to medieval logicians; it evidently belongs to the family of ‘paradoxes of self-reference’ of which the paradigm is the ‘Liar’ paradox – e.g. consider the version *This proposition is false* which is (apparently!) true if is false and false if it is true.

In the case of (14) we get a paradox by combining the classical definition of validity with self-reference. But exactly how to sort out the tangle is a vexed question!