34 Empty domains?

We stipulated that the domain of quantification for a QL language must contain at least one object. This might seem to be a very modest requirement – we surely don't want to be talking about *nothing*. But we need to comment.

34.1 Dummy names and empty domains

(a) In §32.3, we considered the argument

$$\forall x\, Fx,\ \forall x(Fx \to Gx)\ \therefore\ \forall x\, Gx.$$

And we gave a formal proof warranting the inference here. It starts:

(1)	$\forall x\, Fx$	(Prem)
(2)	$\forall x(Fx \to Gx)$	(Prem)
(3)	Fa	($\forall$E 1)

At line (3), the story goes, we pick an arbitrary member of the domain and dub it with a temporary name. But hold on! *What if the domain is empty?* Then there is nothing to pick out and dub! So our derivation at this point in effect presupposes that the domain is non-empty. In its handling of dummy names, then, at least one of our (entirely standard) QL inference rules presupposes that we are dealing with non-empty domains.

In §27.3 and again in §28.8 we stipulated that domains of quantification for QL languages are always non-empty. We now see that this stipulation is a significant one: it ties in with the natural deduction rules which we have adopted.

(b) Let's explore this link further.

Tachyons are, by definition, physical particles which are superluminal, i.e. which travel faster than the speed of light. Standard physics tells us that such particles would have deeply *weird* properties, like having more energy the slower they go. Adopting a QL language quantifying over physical particles, and with the obvious interpretations of the predicates, the following is therefore true:

(1) $\forall x(Tx \to Wx)$.

Now, given this truth, we of course *can't* use our QL inference rules to deduce

(2) $\exists x\, Wx$.

Which is just as it should be. We don't want the truth (1) – *if* something is a tachyon, *then* it has the weird property of having more energy at slower speeds – to entail a proposition that asserts that there really *are* some weird particles of this kind. So far so good.

But now consider what happens if – perhaps in the spirit of Quine's maxim of shallow analysis from §33.3(d) – we think along the following lines. The current topic is tachyons, so can't we make things simpler by taking our domain for now to be just them? (Compare our handling of argument **F** in the previous chapter where we kept things simple by taking the domain to be just students in the logic class.) Using a formal language where the quantifiers run over just tachyons, the uncontentious claim that tachyons are weird is then regimented as

(1′) $\forall \mathsf{x}\, \mathsf{Wx}$.

However, using our adopted natural deduction rules, we get the following proof **A**:

(1′)	$\forall \mathsf{x}\, \mathsf{Wx}$	(Prem)
(2′)	Wa	($\forall$E 1′)
(3′)	$\exists \mathsf{x}\, \mathsf{Wx}$	($\exists$I 2′)

Yet (3′) is an existential claim, only true if there *is* something which has the weird property of having more energy at slower speeds. So we seem to have deduced the existence of something truly weird from a reformulation of what was originally supposed to be an uncontroversial bit of physics.

What has gone wrong here? Again, at the second step of **A**, we take an arbitrary member of the current domain and dub it. *But we can only do that if the domain is populated.* So, in taking the domain of the relevant language to comprise just tachyons, but also assuming that we can apply our current logical apparatus, *we are already implicitly assuming that there are tachyons.*

(c) Suppose the domain of the relevant QL language is empty. Then both $\exists\xi\alpha(\xi)$ and $\exists\xi\neg\alpha(\xi)$ will always be trivially false (since nothing exists in the domain). And so the negation of the second, i.e. $\forall\xi\alpha(\xi)$, will be trivially true. (You can also look at it this way: $\forall\xi\alpha(\xi)$ is always true in the empty domain, because it is true that *if* a thing is in the domain then it satisfies the condition expressed by α – the antecendent of this generalized conditional is always false.) So, the pattern of reasoning in a proof taking us from $\forall\xi\alpha(\xi)$ to $\exists\xi\alpha(\xi)$ will take us from truth to falsehood if applied in an empty domain. Which is what happens in **A**. There are no such things as tachyons. So the stipulated domain of quantification is empty. And that makes (1′) is true and (3′) false.

(d) For a second line of argument that breaks down in empty domains, consider **B**:

(1)		
(2)	$(\mathsf{Fa} \vee \neg\mathsf{Fa})$	(LEM)
(3)	$\exists \mathsf{x}(\mathsf{Fx} \vee \neg\mathsf{Fx})$	($\exists$I 2)

Our rules allow us to prove any instance of the Law of Excluded Middle from no premisses. Assume we are dealing with a QL language which has the unary predicate F (expressing some property F): then in particular we will be able to prove (2) from no premisses. And then (3) follows by (∃I). However (3) is another existential claim; it tells us that there is something in the domain which is either F or not F. And that can only be true if there *is* something in the domain!

34.2 Preserving standard logic

Proof **A** illustrates that derivations in our natural deduction system are not always truth-preserving, *if* we allow empty domains. Proof **B** illustrates that theorems of our system are not always true, *if* we allow empty domains.

How should we respond to this observation? Some logicians argue:

> An inference is logically valid if it is necessarily truth-preserving in virtue of topic-neutral features of its structure. And formal logic is the study of logical validity, using regimented languages to enable us to bring out how arguments of certain forms are valid irrespective of their subject-matter.
>
> Now, sometimes we want to argue logically about the properties of things which we already know to exist (electrons, say). Other times we want to argue in an exploratory way, in ignorance of whether what we are talking about exists (superstrings, perhaps). While sometimes we want to argue about things that we believe don't exist, precisely in order to try to show that they don't exist (tachyons, perhaps). And we presumably want to regiment correct forms of inference which we can apply neutrally across these different cases. Hence *one* way our formal logic should be topic-neutral is by allowing empty domains. But our current QL rules – being incorrect for empty domains – are not topic-neutral. So they don't correctly capture only logical validities and logical truths. Therefore our natural deduction proof system needs revision.

Persuaded by such reasoning, some logicians do advocate the general adoption of a *free* logic – i.e. a logic free of existence assumptions, allowing empty domains (and often also allowing 'empty proper names' which don't have a reference).

So how might the defender of our standard QL logic reply?

> There is no One True Logic. Choosing a formal logic involves – as we have seen before – weighing up costs and benefits. And the small benefit of having a logic whose inferential principles also hold in empty domains is not worth the cost. After all, when we want to argue about things that do not/might not exist, we already have sufficient resources while still using standard logic.
>
> First, a suitably inclusive wider domain is usually easily found (indeed, will typically be in play when engaged in serious inquiry rather than concocting artificial classroom examples). 'If in doubt, go wide' is a good motto – and see again §30.3. For example, instead of taking the domain to be tachyons and regimenting the proposition that all tachyons are weird as

$\forall x\, Wx$, we can more naturally take the domain more inclusively to be, say, physical particles. We can then regiment that proposition as we initially did, as $\forall x(Tx \rightarrow Wx)$ and lose the unwanted inference to $\exists x\, Wx$.

Second, if we continue to have lingering doubts about some more inclusive domain, we can (and do) proceed in an exploratory, non-committal, suppositional mode. For example, consider mathematical inquiry which proceeds in the supposedly all-inclusive framework of set* theory. What if we are sceptical about (some) sets*? We can bracket our set-theoretic investigations with an unspoken 'Ok, let's take it, for the sake of argument, that there *is* this wildly infinitary universe that standard set* theory talks about ...'. And then, within the scope of that bracketing assumption, we plunge in and quantify over sets* in the usual way, and continue our explorations *as if* we are dealing with a suitably populated domain, to see where our investigations get to.

So yes, once we have made the supposition for the sake of further exploration that there are sets* or superstrings or whatever, we might want the same logic to apply in each case, topic-neutrally. But there is no need for this logic we use, once we are working within the scope of the supposition that we *are* talking about something, to still remain neutral about whether there is anything in the domain.

The debate, predictably, will continue. But we have perhaps said enough to explain why, at least for our introductory purposes, it is defensible to stick with our standard logic, the logic for reasoning about non-empty domains.

34.3 Summary

Our standard natural deduction rules presuppose that we are dealing with non-empty domains.

We could revise our logic to give rules for reasoning that apply to populated and empty domains alike. But given that we almost always reason while presupposing – if only for the sake of argument – that we are indeed talking about something, the standard rules certainly apply widely enough to warrant continuing to explore them.