Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 20

(a) Show that the following inferences (in suitable languages, of course) can be warranted by proofs using our rules for conjunction and negation:

(1) $\mathsf{P},\ \mathsf{Q},\ \mathsf{R}\ \therefore\ (\mathsf{P} \land (\mathsf{P} \land (\mathsf{Q} \land \mathsf{Q})))$

(2) $(\mathsf{P} \land (\mathsf{Q} \land \mathsf{R}))\ \therefore\ ((\mathsf{Q} \land \neg\neg\mathsf{P}) \land \mathsf{R})$

(3) $(\mathsf{P} \land \mathsf{Q}),\ \neg(\mathsf{P} \land \mathsf{R}),\ \neg(\mathsf{Q} \land \mathsf{S})\ \therefore\ (\neg\mathsf{R} \land \neg\mathsf{S})$

(4) $\neg(\mathsf{P} \land \neg\mathsf{Q}),\ \neg(\mathsf{Q} \land \neg\neg\mathsf{R})\ \therefore\ \neg(\mathsf{P} \land \neg\neg\mathsf{R})$

(5) $\neg((\mathsf{P} \lor \mathsf{R}) \land \neg\neg(\neg\mathsf{S} \land \mathsf{Q})),\ \neg\neg(\neg\mathsf{S} \land \mathsf{Q})\ \therefore\ \neg(\mathsf{P} \lor \mathsf{R})$

(6) $\neg(\mathsf{P} \land \mathsf{S}),\ \neg(\neg\mathsf{S} \land \mathsf{Q})\ \therefore\ \neg((\mathsf{P} \land \mathsf{R}) \land \mathsf{Q})$

(7) $\neg(\mathsf{P} \land \neg(\mathsf{S} \land \mathsf{Q})),\ (\neg\mathsf{R} \land \neg\neg\mathsf{P})\ \therefore\ (\mathsf{Q} \land \neg\neg\neg\mathsf{R})$

(8) $\neg(\mathsf{P} \land \mathsf{S}),\ \neg(\neg\mathsf{S} \land \mathsf{Q}),\ ((\mathsf{P} \land \mathsf{R}) \land \mathsf{Q})\ \therefore\ \mathsf{P}'$

(9) $\neg(\mathsf{P} \land \neg\neg\neg\bot)\ \therefore\ \neg\mathsf{P}$

(10) $(\mathsf{P} \to \mathsf{Q})\ \therefore ((\mathsf{P} \to \mathsf{Q})\ \land \neg\bot)$

(b*) Recall the 'Γ' notation from Exercises 16(c*), introduced to indicate some wffs (zero, one, or many), with 'Γ, α' indicating those wffs together with α. And recall the use of 'iff' introduced in §18.6. We now add a new pair of definitions

> Γ are *S-consistent* – i.e., are consistent as far as the proof system S can tell – iff there is no proof in system S of $\bot$ from Γ as premisses.
>
> Γ are *S-inconsistent* iff there is an S-proof of $\bot$ from Γ as premisses.

Let S be the current proof system with our conjunction and negation rules. Show:

(1) α can be derived in S from Γ as premisses iff $\Gamma, \neg\alpha$ are S-inconsistent.

Now for three results (for eventual use in the Appendix) about what we can *add* to S-consistent wffs while keeping them S-consistent. First, note that if Γ, α are S-inconsistent, Γ proves $\neg\alpha$; so if Γ, α are S-inconsistent and $\neg\neg\alpha$ is one of the wffs Γ, then Γ must already be S-inconsistent. (Explain why!) Conclude that

(2) If the wffs Γ are S-consistent and $\neg\neg\alpha$ is one of them, then Γ, α are also S-consistent.

We use 'Γ, α, β' to indicate the wffs Γ together with α and β. Show that

(3) If the wffs Γ are S-consistent and $(\alpha \land \beta)$ is one of them, then Γ, α, β are also S-consistent.

Note too that if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent, we can derive both α and β from Γ, and hence can derive $(\alpha \land \beta)$. So if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent and these wffs Γ already include $\neg(\alpha \land \beta)$, then Γ are S-inconsistent (why?). Conclude

(4) If the wffs Γ are S-consistent and $\neg(\alpha \land \beta)$ is one of them, then either $\Gamma, \neg\alpha$ or $\Gamma, \neg\beta$ (or both) are also S-consistent.