Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 32: QL proofs

(a) We said that if $\forall\xi\alpha(\xi)$ is a quantified wff, τ is a term, and $\alpha(\tau)$ is the result of replacing every occurrence of ξ in $\alpha(\xi)$ by τ, then $\alpha(\tau)$ is a wff. Why is that true?

(b) Revisit Exercises **??**, and render the informal arguments given there into suitable QL languages, and then provide formal derivations of the conclusions from the premisses. (Don't skip the propositional reasoning in these easy cases!)

(c) Also translate the following into suitable QL languages, and again provide formal derivations of the conclusions from the premisses:

(1) If Jones is a bad philosopher, then some Welsh speaker is irrational; but every Welsh speaker is rational; hence Jones is not a bad philosopher.
(2) Everyone is such that, if they admire Ludwig, then the world has gone mad. Therefore, if someone admires Ludwig, the world has gone mad.
(3) Jack is taller than Jill. Someone is taller than Jack. If a first person is taller than a second, and the second is taller than a third, then the first person is taller than the third. Hence someone is taller than both Jack and Jill.
(4) Every logician admires Gödel. Whoever admires someone is not without feeling. Hence no logician is without feeling.
(5) Either not everyone liked the cake or someone baked an excellent cake. If I'm right, then whoever bakes an excellent cake ought to be proud. So if everyone liked the cake and I'm right, then someone ought to be proud.

Also translate the following into suitable QL languages and show that they are logical truths by deriving them as theorems:

(6) Everyone is either a logician or not a logician.
(7) It's not the case that all logicians are wise while someone is an unwise logician.
(8) Everyone has someone whom either they love (despite that person loving themself!) or they don't love (unless that person doesn't love themself!).

(d) Give proofs to warrant the following inferences (compare §29.6):

(1) $(\mathsf{P} \land \forall \mathsf{x}\,\mathsf{Fx}) \therefore \forall \mathsf{x}(\mathsf{P} \land \mathsf{Fx})$
(2) $(\exists \mathsf{y}\,\mathsf{Fy} \lor \mathsf{P}) \therefore \exists \mathsf{y}(\mathsf{Fy} \lor \mathsf{P})$
(3) $\forall \mathsf{x}(\mathsf{Fx} \to \mathsf{P}) \therefore (\exists \mathsf{x}\,\mathsf{Fx} \to \mathsf{P})$
(4) $\forall \mathsf{x}(\mathsf{P} \lor \mathsf{Fx}) \therefore (\mathsf{P} \lor \forall \mathsf{x}\,\mathsf{Fx})$
(5) $(\mathsf{P} \to \exists \mathsf{x}\,\mathsf{Fx}) \therefore \exists \mathsf{x}(\mathsf{P} \to \mathsf{Fx})$

(e*) Suppose we set up QL-style languages with only the universal quantifier added to the connectives. Expressions of the form $\exists\xi\alpha(\xi)$ are now introduced into such a language simply as abbreviations for corresponding expressions of the form $\neg\forall\xi\neg\alpha(\xi)$. Show that the two familiar existential quantifier rules would then be derived rules of this system.

Alternatively, suppose we set up a QL-style language with only a 'no' quantifier, expressed using the quantifier-former 'N' (so $\mathsf{N}\xi\alpha(\xi)$ holds when *nothing* satisfies the condition expressed by α). Give introduction and elimination rules for this quantifier. Define the universal and existential quantifier in this new language, and show how to recover their usual inference rules.

(f*) When discussing pairs of PL introduction and elimination rules, we saw that they fitted together in a harmonious way, with the elimination rule as it were reversing or undoing an application of the introduction rule. Can something similar be said about the pairs of QL introduction and elimination rules?