Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 22

Some of the later examples are tricky. So let's repeat the usual mantra: don't *worry too much if you find proof-discovery hard. The important thing, as always, is to make sure you can* understand *the solutions!*

(a) *Warrant the following inferences by PL natural deduction proofs:*

(1) $((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \;\therefore\; (\mathsf{P} \to (\mathsf{Q} \to \mathsf{R}))$

The proof more or less writes itself. The target conclusion is a conditional, so the obvious plan is to assume the antecedent P and aim for the consequent $(\mathsf{Q} \to \mathsf{R})$ (intending to then use (CP)). Our new target in the subproof is *another* conditional; so again we assume its antecedent Q and aim for its consequent R (preparing to invoke (CP) again). Filling in the details we get

(1)	$((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$	(Prem)
(2)	$\quad \mathsf{P}$	(Supp)
(3)	$\quad\quad \mathsf{Q}$	(Supp)
(4)	$\quad\quad (\mathsf{P} \land \mathsf{Q})$	($\land$I 2, 3)
(5)	$\quad\quad \mathsf{R}$	(MP 4, 1)
(6)	$\quad (\mathsf{Q} \to \mathsf{R})$	(CP 3–5)
(7)	$(\mathsf{P} \to (\mathsf{Q} \to \mathsf{R}))$	(CP 2–6)

(2) $(\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \;\therefore\; ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$

This is even more straightforward. We want to prove a conditional: so make the default move of assuming the antecedent and deriving the consequent and then applying (CP):

(1)	$(\mathsf{P} \to (\mathsf{Q} \to \mathsf{R}))$	(Prem)
(2)	$\quad (\mathsf{P} \land \mathsf{Q})$	(Supp)
(3)	$\quad \mathsf{P}$	($\land$E 2)
(4)	$\quad \mathsf{Q}$	($\land$E 2)
(5)	$\quad (\mathsf{Q} \to \mathsf{R})$	(MP 3, 1)
(6)	$\quad \mathsf{R}$	(MP 4, 5)
(7)	$((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$	(CP 2–6)

(3) $((\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R})) \to \bot) \;\therefore\; \neg(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$

Of course, there's nothing significant about the occurrence of the particular subformula $(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$ here. Take *any* wff α, then $(\alpha \to \bot)$ should obviously entail $\neg\alpha$! And it does. The following very simple pattern of reasoning evidently generalizes:

(1)	$((\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R})) \to \bot)$	(Prem)
(2)	$\quad (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	(Supp)
(3)	$\quad \bot$	(MP 2, 1)
(4)	$\neg(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	(RAA 2–3)

(4) $(\mathsf{P} \to \bot),\ (\mathsf{P} \lor \neg\mathsf{Q})\ \therefore\ (\mathsf{Q} \to \bot)$

Again we have a conditional conclusion; so we set out on a subproof, assuming the antecedent and aiming to derive the consequent $\bot$, with a view to using (CP).

And how are we going to derive $\bot$? We will need to argue by cases from the disjunctive second premiss, using ($\lor$E). That is going to mean embedding two more subproofs inside the outer subproof! Which sounds more complicated than it is – the Fitch-style layout as always makes it very clear what is going on:

(1)	$(\mathsf{P} \to \bot)$	(Prem)
(2)	$(\mathsf{P} \lor \neg\mathsf{Q})$	(Prem)
(3)	$\quad \mathsf{Q}$	(Supp)
(4)	$\quad\quad \mathsf{P}$	(Supp)
(5)	$\quad\quad \bot$	(MP 4, 1)
(6)	$\quad\quad \neg\mathsf{Q}$	(Supp)
(7)	$\quad\quad \bot$	(Abs 3, 6)
(8)	$\quad \bot$	($\lor$E 2, 4—5, 6–7)
(9)	$(\mathsf{Q} \to \bot)$	(CP 3–8)

(5) $(\mathsf{P} \land (\neg\mathsf{Q} \to \neg\mathsf{P}))\ \therefore\ (\neg\mathsf{P} \lor (\mathsf{Q} \land \mathsf{P}))$

A general point about proving conclusions of the form $(\alpha \lor \beta)$. It is always worth checking whether you can see a proof of just *one* disjunct; and then the desired conclusion will follow by a final step of $\lor$-introduction.

In this case, the two conjuncts of the premiss, P and $(\neg\mathsf{Q} \to \neg\mathsf{P})$ together give us Q (why?); so we indeed have $(\mathsf{Q} \land \mathsf{P})$ and hence the conclusion. Setting that out, we get

(1)	$(\mathsf{P} \land (\neg\mathsf{Q} \to \neg\mathsf{P}))$	(Prem)
(2)	P	($\land$E, 1)
(3)	$(\neg\mathsf{Q} \to \neg\mathsf{P})$	($\land$E, 1)
(4)	$\quad \neg\mathsf{Q}$	(Supp)
(5)	$\quad \neg\mathsf{P}$	(MP 4, 3)
(6)	$\quad \bot$	(Abs 2, 5)
(7)	$\neg\neg\mathsf{Q}$	(RAA 4–6)
(8)	Q	(DN 7)
(9)	$(\mathsf{Q} \land \mathsf{P})$	($\land$I 8, 2)
(10)	$(\neg\mathsf{P} \lor (\mathsf{Q} \land \mathsf{P}))$	($\lor$I 9)

(6) $((\mathsf{P} \land \mathsf{Q}) \to (\mathsf{Q} \land \mathsf{R})),\ (\mathsf{R} \to \neg\mathsf{P})\ \therefore\ (\mathsf{P} \to \neg\mathsf{Q})$

We want to prove $(\mathsf{P} \to \neg\mathsf{Q})$, so let's start a subproof, assuming P and aiming for $\neg\mathsf{Q}$. And to prove $\neg\mathsf{Q}$ let's do the usual thing when aiming for a negated wff – we will assume Q and aim for contradiction.

So overall, we want a proof shaped like this:

$((\mathsf{P} \wedge \mathsf{Q}) \to (\mathsf{Q} \wedge \mathsf{R}))$	(Prem)
$(\mathsf{R} \to \neg\mathsf{P})$	(Prem)
$\quad \mathsf{P}$	(Supp)
$\quad\quad \mathsf{Q}$	(Supp)
$\quad\quad \vdots$	
$\quad\quad \bot$	
$\quad \neg\mathsf{Q}$	(RAA)
$(\mathsf{P} \to \neg\mathsf{Q})$	(CP)

So how do we fill in the dots?

Our two suppositions taken together give us the antecedent of the first premiss; so we can derive $(\mathsf{Q} \wedge \mathsf{R})$. And *that* together with the second premiss gives us $\neg\mathsf{P}$. And that contradicts our first supposition. Good. We can complete the proof as follows:

(1)	$((\mathsf{P} \wedge \mathsf{Q}) \to (\mathsf{Q} \wedge \mathsf{R}))$	(Prem)
(2)	$(\mathsf{R} \to \neg\mathsf{P})$	(Prem)
(3)	$\quad \mathsf{P}$	(Supp)
(4)	$\quad\quad \mathsf{Q}$	(Supp)
(5)	$\quad\quad (\mathsf{P} \wedge \mathsf{Q})$	($\wedge$I 3, 4)
(6)	$\quad\quad (\mathsf{Q} \wedge \mathsf{R})$	(MP 5, 1)
(7)	$\quad\quad \mathsf{R}$	($\wedge$E 6)
(8)	$\quad\quad \neg\mathsf{P}$	(MP 7, 2)
(9)	$\quad\quad \bot$	(Abs 3, 8)
(10)	$\quad \neg\mathsf{Q}$	(RAA 4–9)
(11)	$(\mathsf{P} \to \neg\mathsf{Q})$	(CP 3–10)

(7) $(\neg\mathsf{S} \to \neg\mathsf{R}),\ ((\mathsf{P} \wedge \mathsf{Q}) \vee \mathsf{R}),\ (\neg\mathsf{S} \to \neg\mathsf{Q})\ \therefore\ (\neg\mathsf{P} \vee \mathsf{S})$

Things are beginning to get a bit more involved. But let's slowly think through the example.

Again we have a disjunctive conclusion. Again a natural thing to ask is whether we can prove either disjunct by itself. It doesn't look as if there is any chance of proving $\neg\mathsf{P}$ (surely the premisses can be true and $\neg\mathsf{P}$ false). So what about proving S?

Well, take the second, disjunctive premiss. The first disjunct gives us Q and hence S using the third premiss (why? – because $\neg\mathsf{S}$ gives us a contradiction). The second disjunct gives us R and hence S using the first premiss (why?). So either way we do get S! And then we can derive $(\neg\mathsf{P} \vee \mathsf{S})$ as wanted.

So let's spell out that line of reasoning:

(1)	$(\neg S \rightarrow \neg R)$	(Prem)
(2)	$((P \wedge Q) \vee R)$	(Prem)
(3)	$(\neg S \rightarrow \neg Q)$	(Prem)
(4)	$\quad (P \wedge Q)$	(Supp)
(5)	$\quad Q$	($\wedge$E 4)
(6)	$\quad\quad \neg S$	(Supp)
(7)	$\quad\quad \neg Q$	(MP 6, 3)
(8)	$\quad\quad \bot$	(Abs 5, 7)
(9)	$\quad \neg\neg S$	(RAA 6–8)
(10)	$\quad S$	(DN 9)
(11)	$\quad R$	(Supp)
(12)	$\quad\quad \neg S$	(Supp)
(13)	$\quad\quad \neg R$	(MP 12, 1)
(14)	$\quad\quad \bot$	(Abs 11, 13)
(15)	$\quad \neg\neg S$	(RAA 12–14)
(16)	$\quad S$	(DN 9)
(17)	S	($\vee$E 2, 4–10, 11–16)
(18)	$(\neg P \vee S)$	($\vee$I 17)

But, as a proof, this is a bit inelegant, as we have repeated the same proof idea twice inside subproofs, making the same supposition twice. So, why not *first* suppose ¬S, before setting out on the subproofs for the ∨-elimination, so we only need to suppose that once (and later only need to apply (DN) once:

(1)	$(\neg S \rightarrow \neg R)$	(Prem)
(2)	$((P \wedge Q) \vee R)$	
(3)	$(\neg S \rightarrow \neg Q)$	(Prem)
(4)	$\quad \neg S$	(Supp)
(5)	$\quad \neg R$	(MP 4, 1)
(6)	$\quad \neg Q$	(MP 4, 3)
(7)	$\quad\quad (P \wedge Q)$	(Supp)
(8)	$\quad\quad Q$	($\wedge$E 4)
(9)	$\quad\quad \bot$	(Abs 8, 6)
(10)	$\quad\quad R$	(Supp)
(11)	$\quad\quad \bot$	(Abs 10, 5)
(12)	$\quad \bot$	($\vee$ 2, 7–9, 10–11)
(13)	$\neg\neg S$	(RAA 4–12)
(14)	S	(DN 13)
(15)	$(\neg P \vee S)$	($\vee$I 14)

(8) $(\neg P \to (Q \wedge R)),\ \neg(R \vee P)\ \therefore\ \neg Q$

Your first thought might well be that there is a mistake here. “The second premiss (by one of De Morgan’s laws) is equivalent to the conjunction of $\neg R$ and $\neg P$, and $\neg P$ combined with the first premiss gives us Q, not $\neg Q$!”

But think again! You are right that the second premiss is equivalent to the conjunction of $\neg R$ and $\neg P$. But $\neg P$ combined with the first premiss gives us not only Q but also R and hence a contradiction. So from these two premisses *anything* follows! Hence there is indeed a proof in our system from the premisses to $\neg Q$. Here’s one:

(1)	$(\neg P \to (Q \wedge R))$	(Prem)
(2)	$\neg(R \vee P)$	(Prem)
(3)	| P	(Supp)
(4)	| $(R \vee P)$	($\vee$I 3)
(5)	| $\bot$	(Abs 4, 2)
(6)	$\neg P$	(RAA 3–5)
(7)	$(Q \wedge R)$	(MP 6, 1)
(8)	R	($\wedge$E 7)
(9)	$(R \vee P)$	($\vee$I 8)
(10)	$\bot$	(Abs 9, 2)
(11)	$\neg Q$	(EFQ 10)

Also give proofs warranting the following inferences:

(9) $Q\ \therefore\ (P \to Q)$

(10) $\neg(P \to Q)\ \therefore\ P$

(11) $\neg(P \to Q)\ \therefore\ \neg Q$

Recall this ‘bookwork’ proof, warranting the inference $(\neg P \vee Q)\ \therefore\ (P \to Q)$:

(1)	$(\neg P \vee Q)$	(Prem)
(2)	| P	(Supp)
(3)	| | $\neg P$	(Supp)
(4)	| | $\bot$	(Abs 2, 3)
(5)	| | Q	(EFQ 4)
(6)	| | Q	(Supp)
(7)	| | Q	(Iter 5)
(8)	| Q	($\vee$E 1, 3–5, 6–7)
(9)	$(P \to Q)$	(CP 2–8)

Abbreviate that line of proof

$(\neg P \vee Q)$	(Prem)
Π	
$(P \to Q)$	

(That ‘Π’ is a capital Greek Pi, so it’s a ‘P’ for ‘proof’!). Then we have the following proofs.

For (9) $\mathsf{Q} \therefore (\mathsf{P} \to \mathsf{Q})$, the following proof will do:

(1)	Q	(Prem)
(2)	$(\neg\mathsf{P} \lor \mathsf{Q})$	($\lor$I 1)
(3)	Π	
(4)	$(\mathsf{P} \to \mathsf{Q})$	

For (10) $\neg(\mathsf{P} \to \mathsf{Q}) \therefore \mathsf{P}$, we have the proof:

(1)	$\neg(\mathsf{P} \to \mathsf{Q})$	(Prem)
(2)	$\quad\mid \neg\mathsf{P}$	(Supp)
(3)	$\quad\mid (\neg\mathsf{P} \lor \mathsf{Q})$	($\lor$I 2)
(4)	$\quad\mid \Pi$	
(5)	$\quad\mid (\mathsf{P} \to \mathsf{Q})$	
(6)	$\quad\mid \bot$	(Abs 5, 1)
(7)	$\neg\neg\mathsf{P}$	(RAA 2–6)
(8)	P	(DN 7)

For (11) $\neg(\mathsf{P} \to \mathsf{Q}) \therefore \neg\mathsf{Q}$, we have the proof:

(1)	$\neg(\mathsf{P} \to \mathsf{Q})$	(Prem)
(2)	$\quad\mid \mathsf{Q}$	(Supp)
(3)	$\quad\mid (\neg\mathsf{P} \lor \mathsf{Q})$	($\lor$I 2)
(4)	$\quad\mid \Pi$	
(5)	$\quad\mid (\mathsf{P} \to \mathsf{Q})$	
(6)	$\quad\mid \bot$	(Abs 5, 1)
(7)	$\neg\mathsf{Q}$	(RAA 2–6)

So far so good. But can we proceed more directly, without going via the bookwork proof Π? For (9) we can try looking for a proof of the following shape:

Q	(Prem)
$\quad\mid \mathsf{P}$	(Supp)
$\quad\mid \vdots$	
$\quad\mid \mathsf{Q}$	
$(\mathsf{P} \to \mathsf{Q})$	(CP)

But how can we fill in the dots? We can use a little device like this:

(1)	Q	(Prem)
(2)	$\quad\mid \mathsf{P}$	(Supp)
(3)	$\quad\mid (\mathsf{P} \land \mathsf{Q})$	($\land$E 2, 1)
(4)	$\quad\mid \mathsf{Q}$	($\land$E 3)
(5)	$(\mathsf{P} \to \mathsf{Q})$	(CP 2–4)

However, we don't even need to do that – for remember we have the iteration rule available, and the following is a legitimately formed derivation in our proof-system:

(1)	$\mid$ Q	(Prem)
(2)	$\mid$ $\mid$ P	(Supp)
(3)	$\mid$ $\mid$ Q	(Iter 1)
(4)	$\mid$ $(P \to Q)$	(CP 2–3)

(If you are a bit suspicious about the use of (CP) here – although it does conform to our statement of the rule in *IFL2* – we discuss this sort of proof in the book in §24.3.)

Next, to get a proof for (10) $\neg(P \to Q) \therefore P$ without appeal to the earlier bookwork proof Π, we want a proof of the following shape

$\mid$ $\neg(P \to Q)$	(Prem)
$\mid$ $\mid$ $\neg P$	(Supp)
$\mid$ $\mid$ $\vdots$	
$\mid$ $\mid$ $(P \to Q)$	
$\mid$ $\mid$ $\bot$	
$\mid$ $\neg\neg P$	(RAA)
$\mid$ P	(DN)

So how do we fill in the dots? Given we want to derive $(P \to Q)$ in the subproof, we do the obvious thing – assume P and aim for Q. And that's easy!

(1)	$\mid$ $\neg(P \to Q)$	(Prem)
(2)	$\mid$ $\mid$ $\neg P$	(Supp)
(3)	$\mid$ $\mid$ $\mid$ P	(Supp)
(4)	$\mid$ $\mid$ $\mid$ $\bot$	(Abs 3,2)
(5)	$\mid$ $\mid$ $\mid$ Q	(EFQ 4)
(6)	$\mid$ $\mid$ $(P \to Q)$	(CP 3–5)
(7)	$\mid$ $\mid$ $\bot$	(Abs 6, 1)
(8)	$\mid$ $\neg\neg P$	(RAA 2–7)
(9)	$\mid$ P	(DN 8)

And for (11) again, note that if we have a proof from α to β we can always turn it into a proof from $\neg\beta$ to $\neg\alpha$ like this:

$\mid$ α	$\Longrightarrow$	$\mid$ $\neg\beta$	
$\mid$ $\vdots$		$\mid$ $\mid$ α	
$\mid$ β		$\mid$ $\mid$ $\vdots$	
		$\mid$ $\mid$ β	
		$\mid$ $\mid$ $\bot$	
		$\mid$ $\neg\alpha$	(RAA)

Hence we can turn our final short proof warranting (9) $Q \therefore (P \to Q)$ into a proof warranting (11) $\neg(P \to Q) \therefore \neg Q$ like this:

(1)	$\neg(P \to Q)$	(Prem)
(2)	Q	(Supp)
(3)	P	(Supp)
(4)	Q	(Iter 2)
(5)	$(P \to Q)$	(CP 3–4)
(6)	$\bot$	(Abs 5, 1)
(7)	$\neg Q$	(RAA 2–6)

Finally then, in this first set of exercises, we want a proof to warrant the inference

(12) $(P \to (Q \vee R)) \ \therefore \ ((P \to Q) \vee (P \to R))$

Suppose the conclusion is *false*. Then each disjunct must be false, so we'd have $(P \to Q)$ false and $(P \to Q)$ false. But that requires P true, and Q and R false, which makes the premiss *false* too. Contraposing, if the premiss is true the conclusion must be true too.

So this inference is valid; but proving it so in our Fitch-style system isn't entirely straightforward.

Evidently, there will be no way of proving the disjunctive conclusion by proving a disjunct. So our only hope is to assume the opposite and aim for a contradiction. So overall, our proof will have the following shape:

$(P \to (Q \vee R))$	(Prem)
$\neg((P \to Q) \vee (P \to R))$	(Supp)
$\vdots$	
$\bot$	
$\neg\neg((P \to Q) \vee (P \to R))$	(RAA)
$((P \to Q) \vee (P \to R))$	(DN)

So how are we going to fill in the dots? We can bulldoze through, just doing the "obvious" things at each stage, like this:

(i) Just as we typically use a conjunction by extracting its conjuncts, it is often a good idea to make use of a negated disjunction $\neg(\alpha \vee \beta)$ by extracting the corresponding $\neg\alpha$ and $\neg\beta$. So let's do this to derive $\neg(P \to Q)$ and $\neg(P \to R)$ from our supposition.

(ii) Now we can use proofs as in our answers to exercises (10) and (11) to in turn derive P, $\neg Q$ and $\neg R$ from those two negated conditionals.

(iii) We now have P in play, so we can at last invoke our sole premiss, and use modus ponens to derive $(Q \vee R)$.

(iv) Now show that from $(Q \vee R)$ and $\neg Q$ and $\neg R$ we can derive the desired absurdity, and we are done!

Still, although that proof almost writes itself (especially when we have the answers to exercises (10) and (11) fresh in our minds), it is decidedly long-winded. Can we do better? How else can we fill in the dots of our proof-outline?

We are eventually going to need to use the first premiss in a modus ponens; but to do that we'll need to have its antecedent in play. OK! Let's experiment and see what happens if we *assume* the antecedent for the sake of argument, and derive its consequent:

│ $(P \to (Q \lor R))$	(Prem)
│ │ $\neg((P \to Q) \lor (P \to R))$	(Supp)
│ │ │ P	(Supp)
│ │ │ $(Q \lor R)$	(MP)

So can we usefully deploy this derived disjunction in an argument by cases?? Well, yes:

(i) Q will give us (in fact, by the easy argument we used in our shortest proof for exercise (10)) the conditional $(P \to Q)$; and then we get an obvious contradiction with our supposition. So the other disjunct R holds.

(ii) But if our supposition P gets us to R, then we have a proof of $(P \to R)$ by (CP). And from that we get a contradiction again ....

Putting everything together, we have

(1)	│ $(P \to (Q \lor R))$	(Prem)
(2)	│ │ $\neg((P \to Q) \lor (P \to R))$	(Supp)
(3)	│ │ │ P	(Supp)
(4)	│ │ │ $(Q \lor R)$	(MP 3, 1)
(5)	│ │ │ │ Q	(Supp)
(6)	│ │ │ │ │ P	(Supp)
(7)	│ │ │ │ │ Q	(Iter 5)
(8)	│ │ │ │ $(P \to Q)$	(CP 6–7)
(9)	│ │ │ │ $((P \to Q) \lor (P \to R))$	($\lor$I 8)
(10)	│ │ │ │ $\bot$	(Abs 9, 2)
(11)	│ │ │ │ R	(EFQ 10)
(12)	│ │ │ │ R	(Supp)
(13)	│ │ │ │ R	(Iter 12)
(14)	│ │ │ R	($\lor$E 4, 5–1, 12–13)
(15)	│ │ $(P \to R)$	(CP 3–14)
(16)	│ │ $((P \to Q) \lor (P \to R))$	($\lor$I 15)
(17)	│ │ $\bot$	(Abs 16, 2)
(18)	│ $\neg\neg((P \to Q) \lor (P \to R))$	(RAA 2–17)
(19)	│ $((P \to Q) \lor (P \to R))$	(DN 18)

Kudos if you spotted this proof idea or a close variant!

(b) *Following the general definition in Exercises 20(b*), let's say in particular*

> *Some wffs are PL-consistent if we cannot use premisses from among them to prove $\bot$.*

*In each of the following cases, show that the given wffs are PL-*in*consistent, i.e. show that there* is *a PL proof of absurdity from them as premisses:*

(1) $(\mathsf{P} \to \neg\mathsf{P}),\ (\neg\mathsf{P} \to \mathsf{P})$

Informally, suppose P: then (from the first wff) we can prove ¬P, giving us a contradiction. So ¬P. But that gives us another contradiction (using the second wff)!

(1)	$(\mathsf{P} \to \neg\mathsf{P})$	(Prem)
(2)	$(\neg\mathsf{P} \to \mathsf{P})$	(Prem)
(3)	| P	(Supp)
(4)	| $\neg\mathsf{P}$	(MP 3, 1)
(5)	| $\bot$	(Abs 3, 4)
(6)	$\neg\mathsf{P}$	(RAA 2–5)
(7)	P	(MP 6, 2)
(8)	$\bot$	(Abs 7, 6)

If we use the biconditional symbol to wrap up the two premisses here into a single wff, then we have shown that $(\mathsf{P} \leftrightarrow \neg\mathsf{P})$ entails a contradiction.

(2) $(\neg\mathsf{P} \lor \neg\mathsf{Q}),\ (\mathsf{P} \land \mathsf{Q})$

We' need to use a proof by cases to show that these are inconsistent (for how else could we use the first wff?).

(1)	$(\neg\mathsf{P} \lor \neg\mathsf{Q})$	(Prem)
(2)	$(\mathsf{P} \land \mathsf{Q})$	(Prem)
(3)	| $\neg\mathsf{P}$	(Supp)
(4)	| P	($\land$E 2)
(5)	| $\bot$	(Abs 4, 3)
(6)	| $\neg\mathsf{Q}$	(Supp)
(7)	| Q	($\land$E 2)
(8)	| $\bot$	(Abs 7, 6)
(9)	$\bot$	($\lor$E 1, 3–5, 6–8)

(3) $((\mathsf{P} \to \mathsf{Q}) \land (\mathsf{Q} \to \neg\mathsf{P})),\ (\mathsf{R} \to \mathsf{P}),\ (\neg\mathsf{R} \to \mathsf{P})$

Why are these inconsistent? Well, the two parts of the first premiss together imply ¬P (why? suppose P and we can derive a contradiction). But now put ¬P together with the second and third premisses and modus tollens gives us both ¬R and ¬¬R, another contradiction.

Here is a formal version of this proof idea (except we can go a bit more speedily at the end):

(1)	$((P \to Q) \land (Q \to \neg P))$	(Prem)
(2)	$(R \to P)$	(Prem)
(3)	$(\neg R \to P)$	(Prem)
(4)	$\quad P$	(Supp)
(5)	$\quad (P \to Q)$	($\land$E 1)
(6)	$\quad Q$	(MP 4, 5)
(7)	$\quad (Q \to \neg P)$	($\land$E 1)
(8)	$\quad \neg P$	(MP 6, 7)
(9)	$\quad \bot$	(Abs 4, 8)
(10)	$\neg P$	(RAA 4–9)
(11)	$\quad R$	(Supp)
(12)	$\quad P$	(MP 11, 2)
(13)	$\quad \bot$	(Abs 12, 10)
(14)	$\neg R$	(RAA 11–13)
(15)	P	(MP 14, 3)
(16)	$\bot$	(Abs 15, 10)

(4) $(P \lor (Q \to R)),\ (\neg R \land \neg(P \lor \neg Q))$

Why are this pair inconsistent? Dropping some unnecessary brackets, the first is equivalent to $(P \lor \neg Q \lor R)$ (why?). And by a general version of De Morgan's Law, that is equivalent to $\neg(\neg P \land \neg\neg Q \land \neg R)$ (why?). Which, rearranging, is equivalent to $\neg(\neg R \land \neg(P \lor \neg Q))$ (why?). So indeed the first wff is equivalent to the negation of the second!

To derive a formal contradiction we'll argue by cases from the first wff. P quickly leads to a contradiction with the second wff. To get a contradiction from $(Q \to R)$ and the second wff takes just a bit more work:

(1)	$(P \lor (Q \to R))$	(Prem)
(2)	$(\neg R \land \neg(P \lor \neg Q))$	(Prem)
(3)	$\neg R$	($\land$E 2)
(4)	$\neg(P \lor \neg Q)$	($\land$E 2)
(5)	$\quad P$	(Supp)
(6)	$\quad (P \lor \neg Q)$	($\lor$I 5)
(7)	$\quad \bot$	(Abs 6, 4)
(8)	$\quad (Q \to R)$	(Supp)
(9)	$\quad\quad Q$	(Supp)
(10)	$\quad\quad R$	(MP 8, 7)
(11)	$\quad\quad \bot$	(Abs 10, 3)
(12)	$\quad \neg Q$	(RAA 9–11)
(13)	$\quad (P \lor \neg Q)$	($\lor$I 12)
(14)	$\quad \bot$	(Abs 13, 4)
(15)	$\bot$	($\lor$E 1, 5–7, 8–14)

(5) $(\neg P \vee R)$, $\neg(R \vee S)$, $(P \vee Q)$, $\neg(Q \wedge \neg S)$.

Let's use an argument from cases based on the first premiss:

$(\neg P \vee R)$	(Prem)
$\neg(R \vee S)$	(Prem)
$(P \vee Q)$	(Prem)
$\neg(Q \wedge \neg S)$	(Prem)
$\quad \neg P$	(Supp)
$\quad \vdots$	
$\quad \bot$	
$\quad R$	(Supp)
$\quad \vdots$	
$\quad \bot$	
$\bot$	(∨E)

To fill in the second gap, note that R implies $(R \vee S)$, contradicting the second premiss. To fill in the first gap, note that $\neg P$ and the third premiss give us Q by disjunctive syllogism; and then Q plus the forth premiss give us S and hence $(R \vee S)$, again contradicting the second premiss. In detail:

(1)	$(\neg P \vee R)$	(Prem)
(2)	$\neg(R \vee S)$	(Prem)
(3)	$(P \vee Q)$	(Prem)
(4)	$\neg(Q \wedge \neg S)$	(Prem)
(5)	$\quad \neg P$	(Supp)
(6)	$\quad\quad P$	(Supp)
(7)	$\quad\quad \bot$	(Abs 6, 5)
(8)	$\quad\quad Q$	(EFQ 7)
(9)	$\quad\quad Q$	(Supp)
(10)	$\quad\quad Q$	(Iter 9)
(11)	$\quad Q$	(∨E 3 6–8, 9–10)
(12)	$\quad\quad \neg S$	(Supp)
(13)	$\quad\quad (Q \wedge \neg S)$	(∧I 11, 12)
(14)	$\quad\quad \bot$	(Abs 13, 4)
(15)	$\quad \neg\neg S$	(RAA 12–14)
(16)	$\quad S$	(DN 15)
(17)	$\quad (R \vee S)$	(∨I 17)
(18)	$\quad \bot$	(Abs 17, 2)
(19)	$\quad R$	(Supp)
(20)	$\quad (R \vee S)$	(∨I 19)
(21)	$\quad \bot$	(Abs 20, 2)
(22)	$\bot$	(∨E 1, 5–18, 19–21)

We could, of course, equally well have used an argument whose overall structure is an argument from cases based on the third premiss. Fill in such an argument. Maybe you can get a slightly shorter proof!

(c) *Suppose that we use '$\leftrightarrow$' so that an expression of the form* $(\alpha \leftrightarrow \gamma)$ *is simply an* abbreviation *of the corresponding expression of the form* $((\alpha \to \gamma) \wedge (\gamma \to \alpha))$. *Warrant the following inferences by* PL *natural deduction proofs:*

(1) $(\mathsf{P} \leftrightarrow \mathsf{Q}) \;\therefore\; (\mathsf{Q} \leftrightarrow \mathsf{P})$
(2) $(\mathsf{P} \leftrightarrow \mathsf{Q}) \;\therefore\; (\neg\mathsf{P} \leftrightarrow \neg\mathsf{Q})$
(3) $(\mathsf{P} \leftrightarrow \mathsf{Q}),\ (\mathsf{Q} \leftrightarrow \mathsf{R}) \;\therefore\; (\mathsf{P} \leftrightarrow \mathsf{R})$

Let's use '(Unabbrev)' to indicate the unpacking of a biconditional abbreviation into the conjunction of two one-way conditionals, and '(Abbrev)' to indicate abbreviating the conjunction of a conditional and its converse into a biconditional.

Then we have the following three easy enough proofs. In each case, we just unpack the premisses; do a proof of each of the two conditionals packed into the conclusion (the proofs will be similar), and then combine the results.

(1)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	$((\mathsf{P} \to \mathsf{Q}) \wedge (\mathsf{Q} \to \mathsf{P}))$	(Unabbrev 1)
(3)	$(\mathsf{Q} \to \mathsf{P})$	($\wedge$E 2)
(4)	$(\mathsf{P} \to \mathsf{Q})$	($\wedge$E 2)
(5)	$((\mathsf{Q} \to \mathsf{P}) \wedge (\mathsf{P} \to \mathsf{Q}))$	($\wedge$I 3, 4)
(6)	$(\mathsf{Q} \leftrightarrow \mathsf{P})$	(Abbrev 5)

(1)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	$((\mathsf{P} \to \mathsf{Q}) \wedge (\mathsf{Q} \to \mathsf{P}))$	(Unabbrev 1)
(3)	$\quad \neg\mathsf{P}$	(Supp)
(4)	$\quad\quad \mathsf{Q}$	(Supp)
(5)	$\quad\quad (\mathsf{Q} \to \mathsf{P})$	($\wedge$E 2)
(6)	$\quad\quad \mathsf{P}$	(MP 4, 5)
(7)	$\quad\quad \bot$	(Abs 6, 3)
(8)	$\quad \neg\mathsf{Q}$	(RAA 4–7)
(9)	$(\neg\mathsf{P} \to \neg\mathsf{Q})$	(CP 3–8)
(10)	$\quad \neg\mathsf{Q}$	(Supp)
(11)	$\quad\quad \mathsf{P}$	(Supp)
(12)	$\quad\quad (\mathsf{P} \to \mathsf{Q})$	($\wedge$E 2)
(13)	$\quad\quad \mathsf{Q}$	(MP 11,12)
(14)	$\quad\quad \bot$	(Abs 10, 13)
(15)	$\quad \neg\mathsf{P}$	(RAA 11–14)
(16)	$(\neg\mathsf{Q} \to \neg\mathsf{P})$	(CP 10–15)
(17)	$((\neg\mathsf{P} \to \neg\mathsf{Q}) \wedge (\neg\mathsf{Q} \to \neg\mathsf{P}))$	($\wedge$I 9, 16)
(18)	$(\neg\mathsf{P} \leftrightarrow \mathsf{Q})$	(Abbrev 17)

(1)	$(P \leftrightarrow Q)$	(Prem)
(2)	$(Q \leftrightarrow R)$	(Prem)
(3)	$((P \rightarrow Q) \land (Q \rightarrow P))$	(Unabbrev 1)
(4)	$((Q \rightarrow R) \land (R \rightarrow Q))$	(Unabbrev 2)
(5)	$\quad P$	(Supp)
(6)	$\quad (P \rightarrow Q)$	($\land$E 3)
(7)	$\quad Q$	(MP 5, 6)
(8)	$\quad (Q \rightarrow R)$	($\land$E 4)
(9)	$\quad R$	(MP 7, 8)
(10)	$(P \rightarrow R)$	(CP 5–9)
(11)	$\quad R$	(Supp)
(12)	$\quad (R \rightarrow Q)$	($\land$E 4)
(13)	$\quad Q$	(MP 11, 12)
(14)	$\quad (Q \rightarrow P)$	($\land$E 3)
(15)	$\quad P$	(MP 13, 14)
(16)	$(R \rightarrow P)$	(CP 11–15)
(17)	$((P \rightarrow R) \land (R \rightarrow P))$	($\land$I 3, 4)
(18)	$(P \leftrightarrow R)$	(Abbrev 5)

Suppose alternatively that we introduce '$\leftrightarrow$' to PL as a basic built-in biconditional connective. Give introduction and elimination rules for this new connective (rules which, like the rules for '$\rightarrow$', don't *mention any other connective). Use these new rules to warrant (1) to (3) again.*

The introduction rule for the one-way conditional is (CP), Conditional Proof which calls on a subproof from the antecedent to the consequent. So the obvious introduction rule for the two-way conditional will call on a *pair* of subproofs, one for each direction:

> *$\leftrightarrow$-introduction* Given a finished subproof starting with the temporary supposition α and ending β and also a finished subproof starting with the temporary supposition β and ending α, we can derive $(\alpha \leftrightarrow \beta)$.

Similarly, the elimination rule for the one-way conditional is (MP), Modus Ponens, which says that given the antecedent of a conditional we can derive its consequent. So the elimination rule for the two-way conditional will be a symmetric version of that: from one component of a conditional we can derive the other:

> *$\leftrightarrow$-elimination* Given $(\alpha \leftrightarrow \beta)$ and α, we can derive β. Equally, given $(\alpha \leftrightarrow \beta)$ and β, we can derive α.

If you find a diagrammatic presentation helpful, we can display the rules like this (as with the familiar ($\land$E), our new ($\leftrightarrow$E) is a pair of rules, and the inputs can appear in either order; and as with ($\lor$E), the two subproofs needed for an application of ($\leftrightarrow$I) can in principle appear in either order too):

$$
(\leftrightarrow\text{E}) \qquad
\begin{array}{c}
(\alpha \to \beta)\\
\vdots\\
\alpha\\
\vdots\\
\beta
\end{array}
\qquad
\begin{array}{c}
(\alpha \to \beta)\\
\vdots\\
\beta\\
\vdots\\
\alpha
\end{array}
\qquad\qquad
(\leftrightarrow\text{I}) \qquad
\begin{array}{l}
\;|\;\underline{\alpha}\\
\;|\;\;\vdots\\
\;|\;\beta\\
\;|\;\underline{\beta}\\
\;|\;\;\vdots\\
\;|\;\alpha\\
(\alpha \leftrightarrow \beta)
\end{array}
$$

Using these new rules, we have new proofs warranting the inferences (1) to (3), using the predictable layout and annotation (as with (∨E), we will help the eye by separating two subproofs by a little horizontal bar when one immediately follows the other):

(1)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	Q	(Supp)
(3)	P	(↔E 2, 1)
(4)	P	(Supp)
(5)	Q	(↔E 4, 1)
(6)	$(\mathsf{Q} \leftrightarrow \mathsf{P})$	(↔I 2–3, 4–5)

(1)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	$\neg\mathsf{P}$	(Supp)
(3)	Q	(Supp)
(4)	P	(↔E 3, 1)
(5)	$\bot$	(Abs 4, 2)
(6)	$\neg\mathsf{Q}$	(RAA 3–5)
(7)	$\neg\mathsf{Q}$	(Supp)
(8)	P	(Supp)
(9)	Q	(↔E 8, 1)
(10)	$\bot$	(Abs 7, 9)
(11)	$\neg\mathsf{P}$	(RAA 8–10)
(12)	$(\neg\mathsf{P} \leftrightarrow \mathsf{Q})$	(↔I 2–6, 7–11)

(1)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	$(\mathsf{Q} \leftrightarrow \mathsf{R})$	(Prem)
(3)	P	(Supp)
(4)	Q	(↔E 3, 1)
(5)	R	(↔E 4, 2)
(6)	R	(Supp)
(7)	Q	(↔E 6, 2)
(8)	P	(↔E 7, 1)
(9)	$(\mathsf{P} \leftrightarrow \mathsf{R})$	(↔I 3–5, 6–8)

Also give proofs to warrant the following:

(4) $\mathsf{P},\ \mathsf{Q}\ \therefore\ (\mathsf{P} \leftrightarrow \mathsf{Q})$
(5) $\neg(\mathsf{P} \leftrightarrow \mathsf{Q})\ \therefore\ ((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$
(6) $(\mathsf{P} \leftrightarrow \mathsf{R}),\ (\mathsf{Q} \leftrightarrow \mathsf{S})\ \therefore\ ((\mathsf{P} \vee \mathsf{Q}) \leftrightarrow (\mathsf{R} \vee \mathsf{S}))$

The first of these reflects that we are dealing with a *material* biconditional – it is enough for the truth of a biconditional that both components are true. The quick proof is this:

(1)	P	(Prem)
(2)	Q	(Prem)
(3)	$\quad \mathsf{P}$	(Supp)
(4)	$\quad \mathsf{Q}$	(Iter 2)
(5)	$\quad \mathsf{Q}$	(Supp)
(6)	$\quad \mathsf{P}$	(Iter 1)
(7)	$(\mathsf{P} \leftrightarrow \mathsf{R})$	($\leftrightarrow$I 3–4, 5–6)

(5) has a disjunctive conclusion and evidently we can't prove it by proving just one of the disjuncts. So our best hope is to assume the conclusion is false and aim for a contradiction. And how do we get a contradiction with the original premiss? – obviously enough, by proving $(\mathsf{P} \leftrightarrow \mathsf{Q})$! And how do we do *that*? By providing a subproof from P to Q and a subproof from Q to P.

So we might hope for a proof with the following overall shape:

$\neg(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
$\quad \neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(Supp)
$\quad\quad \mathsf{P}$	(Supp)
$\quad\quad \vdots$	
$\quad\quad \mathsf{Q}$	
$\quad\quad \mathsf{Q}$	(Supp)
$\quad\quad \vdots$	
$\quad\quad \mathsf{P}$	
$\quad (\mathsf{P} \leftrightarrow \mathsf{Q})$	($\leftrightarrow$I)
$\quad \bot$	(Abs)
$\neg\neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(RAA)
$((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(DN)

So how do we fill in the dots?

We need to argue, in the first case, from our supposition P to Q. Well, the obvious trick is to suppose $\neg\mathsf{Q}$ and aim for a reductio. And then, lo and behold, given that new supposition, we would have $(\mathsf{P} \wedge \neg\mathsf{Q})$; and from that a step of $\vee$-introduction gives us a contradiction with the first of our suppositions!

The second case is of course similar, and the completed proof will look like this:

(1)	$\neg(\mathsf{P} \leftrightarrow \mathsf{Q})$	(Prem)
(2)	$\neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(Supp)
(3)	P	(Supp)
(4)	$\neg\mathsf{Q}$	(Supp)
(5)	$(\mathsf{P} \wedge \neg\mathsf{Q})$	($\wedge$I 3, 4)
(6)	$((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	($\vee$I 5)
(7)	$\bot$	(Abs 6, 2)
(8)	$\neg\neg\mathsf{Q}$	(RAA 3–7)
(9)	Q	(DN 8)
(10)	Q	(Supp)
(11)	$\neg\mathsf{P}$	(Supp)
(12)	$(\neg\mathsf{P} \wedge \mathsf{Q})$	($\wedge$I 11, 10)
(13)	$((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	($\vee$I 12)
(14)	$\bot$	(Abs 13, 2)
(15)	$\neg\neg\mathsf{P}$	(RAA 11–14)
(16)	P	(DN 15)
(17)	$(\mathsf{P} \leftrightarrow \mathsf{Q})$	($\leftrightarrow$I 3–9, 10–16)
(18)	$\bot$	(Abs 17, 1)
(19)	$\neg\neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(RAA 2–18)
(20)	$((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\neg\mathsf{P} \wedge \mathsf{Q}))$	(DN 19)

And for (6) we can argue by cases (twice!) like this:

(1)	$(\mathsf{P} \leftrightarrow \mathsf{R})$	(Prem)
(2)	$(\mathsf{Q} \leftrightarrow \mathsf{S})$	(Prem)
(3)	$(\mathsf{P} \vee \mathsf{Q})$	(Supp)
(4)	P	(Supp)
(5)	R	($\leftrightarrow$E 4, 1)
(6)	$(\mathsf{R} \vee \mathsf{S})$	($\vee$I 5)
(7)	Q	(Supp)
(8)	S	($\leftrightarrow$E 7, 2)
(9)	$(\mathsf{R} \vee \mathsf{S})$	($\vee$I 8)
(10)	$(\mathsf{R} \vee \mathsf{S})$	($\vee$E 3, 4–6, 7–9)
(11)	$(\mathsf{R} \vee \mathsf{S})$	(Supp)
(12)	R	(Supp)
(13)	P	($\leftrightarrow$E 12 1)
(14)	$(\mathsf{P} \vee \mathsf{Q})$	($\vee$I 14)
(15)	S	(Supp)
(16)	Q	($\leftrightarrow$E 15, 2)
(17)	$(\mathsf{P} \vee \mathsf{Q})$	($\vee$I 16)
(18)	$(\mathsf{P} \vee \mathsf{Q})$	($\vee$E 11, 12-14, 15–17)
(19)	$((\mathsf{P} \vee \mathsf{Q}) \rightarrow (\mathsf{R} \vee \mathsf{S}))$	($\leftrightarrow$I 3–10, 11–18)

Use a truth-table to confirm that the following wffs are tautologically equivalent:

(7) $(\mathsf{P} \leftrightarrow (\mathsf{Q} \leftrightarrow \mathsf{R})),\ ((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R})$.

For a trickier challenge, outline a proof from the first to the second.

The truth-table is routine, and I won't give it here! The only point in getting you to pause over it is to highlight that these two wffs are indeed equivalent (and do note, the following pair of wffs with one-way conditionals are *not* equivalent! – $(\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})),\ ((\mathsf{P} \to \mathsf{Q}) \to \mathsf{R})$).

As for a formal derivation from the first to the second, it will need overall to be reductio proof of the shape:

$$
\begin{array}{lll}
| & (\mathsf{P} \leftrightarrow (\mathsf{Q} \leftrightarrow \mathsf{R})) & \text{(Prem)} \\
| & \quad | \ \neg((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(Supp)} \\
| & \quad | \ \vdots & \\
| & \quad | \ \bot & \\
| & \neg\neg((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(RAA)} \\
| & ((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(DN)}
\end{array}
$$

Now the supposition is a negated biconditional $\neg(\alpha \leftrightarrow \beta)$ and we know from (5) that from this we can derive a disjunction $((\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta))$. We can then expect to use this disjunction in a proof by cases, hopefully giving us a proof which overall looks like this:

$$
\begin{array}{lll}
| & (\mathsf{P} \leftrightarrow (\mathsf{Q} \leftrightarrow \mathsf{R})) & \text{(Prem)} \\
| & \quad | \ \neg((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(Supp)} \\
| & \quad | \ (((\mathsf{P} \leftrightarrow \mathsf{Q}) \wedge \neg\mathsf{R}) \vee (\neg(\mathsf{P} \leftrightarrow \mathsf{Q}) \wedge \mathsf{R})) & \text{(as in proof for (5))} \\
| & \quad | \quad | \ ((\mathsf{P} \leftrightarrow \mathsf{Q}) \wedge \neg\mathsf{R}) & \text{(Supp)} \\
| & \quad | \quad | \ \vdots & \\
| & \quad | \quad | \ \bot & \\
| & \quad | \quad | \ (\neg(\mathsf{P} \leftrightarrow \mathsf{Q}) \wedge \mathsf{R}) & \text{(Supp)} \\
| & \quad | \quad | \ \vdots & \\
| & \quad | \quad | \ \bot & \\
| & \quad | \ \bot & \text{($\vee$E using disjunction on third line)} \\
| & \neg\neg((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(RAA)} \\
| & ((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R}) & \text{(DN)}
\end{array}
$$

How are we going to fill in the *first* subproof?

(i) We want to be able at some point to make use of the original premiss, so let's see what happens if we suppose P. Then (using the first premiss) we get $(\mathsf{Q} \leftrightarrow \mathsf{R})$. But we also have from the second supposition $(\mathsf{P} \leftrightarrow \mathsf{Q})$. So we get R and hence a contradiction!

(ii) So, still working in the first subproof, we have shown $\neg\mathsf{P}$. And given the first premiss, that yields $\neg(\mathsf{Q} \leftrightarrow \mathsf{R})$ (why?) and hence another disjunction $((\mathsf{Q} \wedge \neg\mathsf{R}) \vee (\neg\mathsf{Q} \wedge \mathsf{R}))$.

(iii) Now we argue by cases again inside the current subproof. The first disjunct $(\mathsf{Q} \wedge \neg\mathsf{R})$ gives us Q. But we still have $(\mathsf{P} \leftrightarrow \mathsf{Q})$ available which gives P and a contradiction since we shown that $\neg\mathsf{P}$. The second disjunct $(\neg\mathsf{Q} \wedge \mathsf{R})$ gives us R which contradicts $\neg\mathsf{R}$ which is still available.

(iv) So, yes, our first subproof ends in absurdity either way!

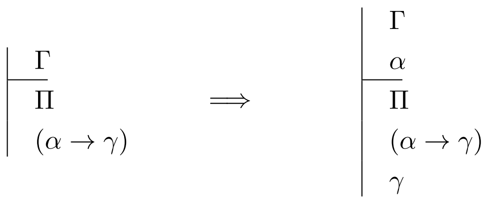
Ouch! That was a bit painful, and we still need to fill in the *second* subproof. But now we've got the hang of things, it will be quite similar, and so let's leave it at that. After all, we only asked for a proof outline!

(d*) *First show*

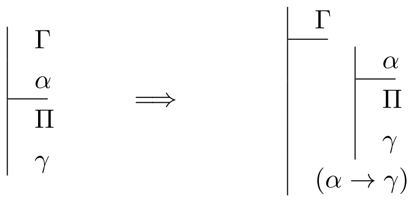
(1) There is a proof of $(\alpha \to \gamma)$ *from the premisses* Γ *if and only if there is a proof of* γ *from* Γ, α*.*

(2) There is a proof of $(\gamma \to \bot)$ *from the premisses* Γ *if and only if there is a proof of* $\neg\gamma$ *from* Γ*.*

For (1), just note that we can turn a proof of $(\alpha \to \gamma)$ from Γ into a proof of γ from α and Γ like this:

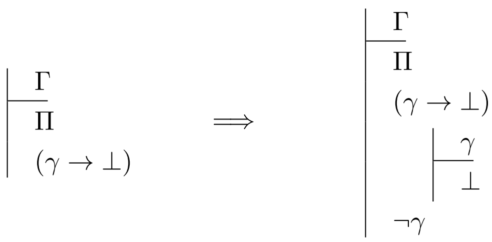


And also conversely we can turn a proof of γ from α and Γ into a proof of $(\alpha \to \gamma)$ from Γ like this:



This second transformation requires just a bit more thought: we need to check that shifting columns like this doesn't stop Π being a well-formed proof. But reflect: everything in Γ remains available for use; and then all the internal relations inside Π stay the same. So, yes, we can still derive γ, setting us up for the final inference by conditional proof.

Similarly, to show (2) note that we can extend a proof $(\gamma \to \bot)$ from Γ to become a proof of $\neg\gamma$ from Γ like this:



And conversely, a proof of $\neg\gamma$ from Γ can be extended to become a proof of $(\gamma \to \bot)$ from Γ like this:

$$\begin{array}{|l} \Gamma \\ \hline \Pi \\ \neg\gamma \end{array} \qquad \Longrightarrow \qquad \begin{array}{|l} \Gamma \\ \hline \Pi \\ \neg\gamma \\ \quad \begin{array}{|l} \gamma \\ \hline \bot \end{array} \\ (\gamma \to \bot) \end{array}$$

So we are done.

And now show the following:

(3) The results of Exercises 21(b) and 22(b*) still obtain when S is the whole PL proof system.*

(4) If Γ are PL-consistent and $(\alpha \to \gamma)$ is one of those wffs, then either $\Gamma, \neg\alpha$ or Γ, γ (or both) are also PL-consistent.

(5) If Γ are PL-consistent and $\neg(\alpha \to \gamma)$ is one of those wffs, then $\Gamma, \alpha, \neg\gamma$ are also PL-consistent.

(3) takes no work. Those earlier results depended only on the availability of the negation, conjunction and disjunction rules, and those rules are still in play.

(4) and (5) are then proved just like those earlier results so we can be brisk. Thus, for (4), we note that if both $\Gamma, \neg\alpha$ and Γ, γ are PL-inconsistent, then we can prove both α and $\neg\gamma$ from Γ (why?). So we can then derive $\neg(\alpha \to \gamma)$ from Γ (why?). Hence, if both $\Gamma, \neg\alpha$ and Γ, γ are PL-inconsistent and Γ also contain $(\alpha \to \gamma)$, then Γ are already PL-inconsistent (why?). Re-arranging gives (4).

For (5), we note that if $\Gamma, \alpha, \neg\gamma$ are PL-inconsistent, then we can derive γ from Γ, α, so then we can go on to derive $(\alpha \to \gamma)$ from Γ. Hence, if $\Gamma, \alpha, \neg\gamma$ are PL-inconsistent and Γ also contain $\neg(\alpha \to \gamma)$, then Γ are already PL-inconsistent. Re-arranging gives (5).