Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 23: PL Theorems

(a) *Show that the following wffs are theorems of our* PL *proof system (in a suitable language).*

(1) $(\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$

This is a wff of the form $(\neg\alpha \lor \alpha)$; so you only have to slightly adjust the 'bookwork' proof for an instance of the Law of Excluded Middle of the form $(\alpha \lor \neg\alpha)$. Thus:

(1)	$\mid\underline{\quad}$	
(2)	$\mid\quad\mid\underline{\neg(\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))}$	(Supp)
(3)	$\mid\quad\mid\quad\mid\underline{((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})}$	(Supp)
(4)	$\mid\quad\mid\quad\mid (\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$	($\lor$I 3)
(5)	$\mid\quad\mid\quad\mid \bot$	(Abs 4, 2)
(6)	$\mid\quad\mid \neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$	(RAA 3–5)
(7)	$\mid\quad\mid (\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$	($\lor$I 6)
(8)	$\mid\quad\mid \bot$	(Abs 7, 2)
(9)	$\mid \neg\neg(\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$	(RAA 2–8)
(10)	$\mid (\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$	(DN 9)

(2) $(((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \to (\mathsf{Q} \to (\mathsf{P} \to \mathsf{R})))$

The proof really does write itself. We need to prove something of the form $(\alpha \to \beta)$ so we assume α and aim for β. But β itself has the form $(\gamma \to \delta)$ so make a new supposition of γ and aim for δ. And δ again is a conditional, namely $(\mathsf{P} \to \mathsf{R})$ – so make a further supposition of its antecedent P and aim for its consequent Q. What else could we sensibly do? So let's do it!

(1)	$\mid\underline{\quad}$	
(2)	$\mid\quad\mid\underline{((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})}$	(Supp)
(3)	$\mid\quad\mid\quad\mid\underline{\mathsf{Q}}$	(Supp)
(4)	$\mid\quad\mid\quad\mid\quad\mid\underline{\mathsf{P}}$	(Supp)
(5)	$\mid\quad\mid\quad\mid\quad\mid (\mathsf{P} \land \mathsf{Q})$	($\land$I 3, 4)
(6)	$\mid\quad\mid\quad\mid\quad\mid \mathsf{R}$	(MP 5, 2)
(7)	$\mid\quad\mid\quad\mid (\mathsf{P} \to \mathsf{R})$	(CP 4–6)
(8)	$\mid\quad\mid (\mathsf{Q} \to (\mathsf{P} \to \mathsf{R}))$	(CP 3–6)
(9)	$\mid (((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \to (\mathsf{Q} \to (\mathsf{P} \to \mathsf{R})))$	(CP 2–8)

(3) $((\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \to ((\mathsf{P} \to \mathsf{Q}) \to (\mathsf{P} \to \mathsf{R})))$

Again, in a similar way, we just need to do the obvious sensible thing at each step, keeping in mind the targets we are aiming at in each subproof!

(1)	$\vert$	
(2)	$\vert\ \vert\ (\mathsf{P} \to (\mathsf{Q} \to \mathsf{R}))$	(Supp)
(3)	$\vert\ \vert\ \vert\ (\mathsf{P} \to \mathsf{Q})$	(Supp)
(4)	$\vert\ \vert\ \vert\ \vert\ \mathsf{P}$	(Supp)
(5)	$\vert\ \vert\ \vert\ \vert\ \mathsf{Q}$	(MP 4, 3)
(6)	$\vert\ \vert\ \vert\ \vert\ (\mathsf{Q} \to \mathsf{R})$	(MP 4, 2)
(7)	$\vert\ \vert\ \vert\ \vert\ \mathsf{R}$	(MP 5, 6)
(8)	$\vert\ \vert\ \vert\ (\mathsf{P} \to \mathsf{R})$	(CP 4–7)
(9)	$\vert\ \vert\ (\mathsf{P} \to \mathsf{Q}) \to (\mathsf{P} \to \mathsf{R}))$	(CP 3–8)
(10)	$\vert\ ((\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \to ((\mathsf{P} \to \mathsf{Q}) \to (\mathsf{P} \to \mathsf{R})))$	(CP 2–8)

(4) $(\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$

We have to prove a disjunction $(\alpha \vee \beta)$. Let's try the frequently useful strategy of assuming $\neg(\alpha \vee \beta)$, using that (as we have done before) to prove both $\neg\alpha$ and $\neg\beta$, and then try to get a contradiction. So let's start:

(1)	$\vert$	
(2)	$\vert\ \vert\ \neg(\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$	(Supp)
(3)	$\vert\ \vert\ \vert\ \neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q}))$	(Supp)
(4)	$\vert\ \vert\ \vert\ (\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$	($\vee$I 3)
(5)	$\vert\ \vert\ \vert\ \bot$	(Abs 4, 2)
(6)	$\vert\ \vert\ \neg\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q}))$	(RAA 3–6)
(7)	$\vert\ \vert\ (\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q}))$	(DN 7)
(8)	$\vert\ \vert\ \vert\ \mathsf{Q}$	(Supp)
(9)	$\vert\ \vert\ \vert\ (\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$	($\vee$I 8)
(10)	$\vert\ \vert\ \vert\ \bot$	(Abs 9, 2)
(11)	$\vert\ \vert\ \neg\mathsf{Q}$	(RAA 8–10)

So now the task is to derive absurdity from (7) and (11). But that's easy! Argue by cases from the disjunction we get from (7):

(12)	$\vert\ \vert\ \mathsf{P}$	($\wedge$E 7)
(13)	$\vert\ \vert\ (\neg\mathsf{P} \vee \mathsf{Q})$	($\wedge$E 7)
(14)	$\vert\ \vert\ \vert\ \neg\mathsf{P}$	(Supp)
(15)	$\vert\ \vert\ \vert\ \bot$	(Abs 12, 14)
(16)	$\vert\ \vert\ \vert\ \mathsf{Q}$	(Supp)
(17)	$\vert\ \vert\ \vert\ \bot$	(Abs 16, 11)
(18)	$\vert\ \vert\ \bot$	($\vee$E 13, 14–15, 16–17)
(19)	$\vert\ \neg\neg(\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$	(RAA 2–18)
(20)	$\vert\ (\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q}))$	(DN 19)

(5) $((\mathsf{P} \to \mathsf{Q}) \vee (\mathsf{Q} \to \mathsf{P}))$

One option is to use (LEM) to give us $(\mathsf{P} \vee \neg\mathsf{P})$ and then argue by cases, deriving the desired disjunct from each of P and $\neg\mathsf{P}$. For, by a now familiar argument, P entails $(\mathsf{Q} \to \mathsf{P})$ and hence the disjunction. While $\neg\mathsf{Q}$ also entails $(\mathsf{Q} \to \mathsf{P})$ and hence the disjunction.

Or we can argue more directly like this (think through the strategy!):

(1)		
(2)	$\quad\neg((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$	(Supp)
(3)	$\quad\quad\mathsf{P}$	(Supp)
(4)	$\quad\quad\quad\neg\mathsf{Q}$	(Supp)
(5)	$\quad\quad\quad\quad\mathsf{Q}$	(Supp)
(6)	$\quad\quad\quad\quad\bot$	(Abs 5, 4)
(7)	$\quad\quad\quad\quad\mathsf{P}$	(EFQ 6)
(8)	$\quad\quad\quad(\mathsf{Q} \to \mathsf{P})$	(CP 5–7)
(9)	$\quad\quad\quad((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$	($\lor$I 8)
(10)	$\quad\quad\quad\bot$	(Abs 9, 2)
(11)	$\quad\quad\neg\neg\mathsf{Q}$	(RAA 4–10)
(12)	$\quad\quad\mathsf{Q}$	(DN 11)
(13)	$\quad(\mathsf{P} \to \mathsf{Q})$	(CP 3–12)
(14)	$\quad((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$	($\lor$I 13)
(15)	$\quad\bot$	(Abs 14, 2)
(16)	$\neg\neg((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$	(RAA 2–15)
(17)	$((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$	(DN 16)

Before moving on, do pause to note the apparent *oddity* in this final wff's being be a theorem. Surely it isn't logically true, for ordinary language conditionals, that if a conditional doesn't hold then its converse does! Question: is this another strike against baldly identifying ordinary language conditionals with material conditionals?

(6) $(((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \to ((\mathsf{P} \to \mathsf{R}) \lor (\mathsf{Q} \to \mathsf{R})))$

Another oddity, you might think, as a theorem. An example from Richard Jeffrey: suppose it is true that if the captain turns his key and the first officer turns *his* key, then the missile fires. Surely it shouldn't follow that either the missile fires if the captain turns his key, or else the missile fires if the first officer turns *his* key. Logic can't rule out failsafe, dual key, systems!

But with the material conditional in play, $((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$ does indeed entail $((\mathsf{P} \to \mathsf{R}) \lor (\mathsf{Q} \to \mathsf{R}))$, and the conditional (6) is indeed a theorem.

Here's one proof strategy.

(i) Assume the antecedent of our conditional, i.e. $((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$; and then assume the negation of its consequent, i.e. assume $\neg((\mathsf{P} \to \mathsf{R}) \lor (\mathsf{Q} \to \mathsf{R}))$, and aim for a contradiction so we can infer the consequent.

(ii) From $\neg((\mathsf{P} \to \mathsf{R}) \lor (\mathsf{Q} \to \mathsf{R}))$, infer both $\neg(\mathsf{P} \to \mathsf{R})$ and $\neg(\mathsf{Q} \to \mathsf{R})$ in the usual way.

(iii) From $\neg(\mathsf{P} \to \mathsf{R})$ infer P and $\neg\mathsf{R}$ (that's a familiar line of proof by now).

(iv) Similarly from $\neg(\mathsf{Q} \to \mathsf{R})$ infer Q (we don't need to infer $\neg\mathsf{R}$ again!).

(v) We have now derived P and Q and so have $(\mathsf{P} \wedge \mathsf{Q})$, and hence have R, by modus ponens with our original assumption.

(vi) So we now have R and $\neg\mathsf{R}$, giving us a contradiction at last!

(vii) So our second supposition $\neg((\mathsf{P} \to \mathsf{R}) \vee (\mathsf{Q} \to \mathsf{R}))$ leads to contradiction so by (RAA) and (DN) we can infer $((\mathsf{P} \to \mathsf{R}) \vee (\mathsf{Q} \to \mathsf{R}))$.

Kudos if you spotted a quicker route!

(7) $(((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P}) \to ((\mathsf{Q} \to \mathsf{P}) \vee \mathsf{P}))$

A slightly tricksy question that is a lot easier than it looks (if you have been paying attention!). For it is standard bookwork to establish an instance of Peirce's Law

$$(((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P}) \ \to \ \mathsf{P})$$

So argue from $((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P})$ as supposition to P in the bookwork way, and then a final trivial step of $(\vee\mathrm{I})$ gives us what we want!

(1)		
(2)	$((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P})$	(Supp)
(3)	$\neg\mathsf{P}$	(Supp)
(4)	P	(Supp)
(5)	$\bot$	(Abs 4, 3)
(6)	Q	(EFQ 5)
(7)	$(\mathsf{P} \to \mathsf{Q})$	(CP 4, 6)
(8)	P	(MP 7, 2)
(9)	$\bot$	(Abs 8, 3)
(10)	$\neg\neg\mathsf{P}$	(RAA 3–9)
(11)	P	(DN 10)
(12)	$((\mathsf{Q} \to \mathsf{P}) \vee \mathsf{P})$	$(\vee\mathrm{I}\ 11)$
(13)	$(((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P}) \to ((\mathsf{Q} \to \mathsf{P}) \vee \mathsf{P}))$	(CP 2–12)

(b*) *More on negation and alternative rules of inference. The following rule is often called* Classical Reductio*, to be carefully distinguished from our (RAA):*

Given a finished subproof starting with the temporary supposition $\neg\alpha$ and concluding $\bot$, we can derive α.

And the following is a form of Peirce's Law *(analogous to (LEM)):*

We can invoke an instance of $((\alpha \to \beta) \to \alpha) \to \alpha)$ at any stage in a proof.

Show that the new proof system which results from our PL proof system by replacing the double negation rule (DN) with either (i) Classical Reductio or (ii) Peirce's Law is equivalent to our current system.

Compare (RAA) on the left and (CR) on the right: crucially, the negation signs are in different places!

$$\text{(RAA)}\qquad \begin{array}{l} \begin{array}{|l} \underline{\alpha} \\ \vdots \\ \bot \end{array} \\ \neg\alpha \end{array} \qquad\qquad\qquad \text{(CR)}\qquad \begin{array}{l} \begin{array}{|l} \underline{\neg\alpha} \\ \vdots \\ \bot \end{array} \\ \alpha \end{array}$$

Now, if we drop (DN) from our proof system, and add (CR) we can still prove everything that we could prove before. Because we can replace a (DN) inference as on the left by a (CR) inference as on the right:

$$\text{(DN)}\qquad \begin{array}{|l} \neg\neg\alpha \\ \vdots \\ \alpha \end{array} \qquad \Longrightarrow \qquad \text{(CR)}\qquad \begin{array}{|l} \neg\neg\alpha \\ \vdots \\ \quad \begin{array}{|l} \underline{\neg\alpha} \\ \bot \end{array} \\ \alpha \end{array}$$

So replacing (DN) by (CR) doesn't *weaken* our system. But it doesn't *strengthen* it either. Because any application of (CR) is equivalent to an application in our system of (RAA) followed by (DN):

$$\text{(CR)}\qquad \begin{array}{l} \begin{array}{|l} \underline{\neg\alpha} \\ \vdots \\ \bot \end{array} \\ \alpha \end{array} \qquad \Longrightarrow \qquad \text{(RAA, DN)}\qquad \begin{array}{l} \begin{array}{|l} \underline{\neg\alpha} \\ \vdots \\ \bot \end{array} \\ \neg\neg\alpha \\ \alpha \end{array}$$

Similarly, if we drop (DN) from our proof system, and add Peirce's Law we can again prove everything that we could prove before. Because we can replace a (DN) inference as on the left by a rather tricksy inference relying on Peirce's Law on the right:

$$\text{(DN)}\qquad \begin{array}{|l} \neg\neg\alpha \\ \vdots \\ \alpha \end{array} \qquad \Longrightarrow \qquad \text{(Peirce, etc.)}\qquad \begin{array}{|ll} \neg\neg\alpha & \\ \vdots & \\ \quad \begin{array}{|l} \underline{(\alpha \to \bot)} \\ \quad \begin{array}{|l} \underline{\alpha} \\ \bot \end{array} \\ \neg\alpha \\ \bot \\ \alpha \end{array} & \begin{array}{l} \text{(Supp)} \\ \text{(Supp)} \\ \text{(MP)} \\ \text{(RAA)} \\ \\ \text{(EFQ)} \end{array} \\ ((\alpha \to \bot) \to \alpha) & \text{(CP)} \\ (((\alpha \to \bot) \to \alpha) \to \alpha) & \text{(Peirce!)} \\ \alpha & \text{(MP)} \end{array}$$

So replacing (DN) by Peirce's Law doesn't *weaken* our system. But it doesn't *strengthen* it either, since we know from the bookwork proof in §23.1 that we can prove instances of Peirce's Law in our current system. So replacing (DN) with (CR) or Peice leaves us with an equivalent system, as claimed.