

## Exercises 32: QL proofs

(a) We said that if  $\forall\xi\alpha(\xi)$  is a quantified wff,  $\tau$  is a term, and  $\alpha(\tau)$  is the result of replacing every occurrence of  $\xi$  in  $\alpha(\xi)$  by  $\tau$ , then  $\alpha(\tau)$  is a wff. Why is that true?

So we don't get our *taus* in a twist, let's very slightly rephrase the question, using  $\tau'$  instead!

We said that if  $\forall\xi\alpha(\xi)$  is a quantified wff,  $\tau'$  is a term, and  $\alpha(\tau')$  is the result of replacing every occurrence of  $\xi$  in  $\alpha(\xi)$  by  $\tau'$ , then  $\alpha(\tau')$  is a wff. Why is that true?

Well, let's ask: how can  $\forall\xi\alpha(\xi)$  be a wff? §28.6 on the syntax of quantifiers gave us the following one and only rule for forming a universally quantified wff:

If  $\alpha(\tau)$  is a QL wff involving one or more occurrences of the term  $\tau$ , and  $\alpha(\xi)$  is the expression which results from replacing the term  $\tau$  throughout  $\alpha(\tau)$  by a variable  $\xi$  new to that wff, then  $\forall\xi\alpha(\xi)$  is a QL wff.

So, given wff  $\forall\xi\alpha(\xi)$  is a wff, there must at least one wff  $\alpha(\tau)$  from which the quantified wff can be formed by our sole rule for building universally quantified wffs.

Take such a  $\alpha(\tau)$ . What happens if you systematically replace  $\tau$  in  $\alpha(\tau)$  by some *other* term  $\tau'$ ? The expression  $\alpha(\tau')$  will be built up from  $\tau'$  (a proper name or dummy name) and other ingredients *exactly* as the wff  $\alpha(\tau)$  is built up from  $\tau$  (another proper name or dummy name) and those same ingredients. So this will make  $\alpha(\tau')$  a wff too. For the status of an expression as a wff obviously doesn't depend on *which* particular proper name or dummy name occurs in a particular place (if one name will do, any other will do too).

Given  $\forall\xi\alpha(\xi)$  can be thought of as built by universally quantifying on  $\tau$  in the instance  $\alpha(\tau)$  – i.e. replacing  $\tau$  with  $\xi$  and prefixing the quantifier  $\forall\xi$  – we can therefore equally well regard the wff  $\forall\xi\alpha(\xi)$  as constructed from the wff (yes, wff!)  $\alpha(\tau')$  by universally quantifying on  $\tau'$  – i.e. replacing  $\tau'$  with  $\xi$  and prefixing the quantifier  $\forall\xi$ !

So, putting that in reverse, we can regard the wff (yes, wff!)  $\alpha(\tau')$  as an instance, the result of result of replacing every occurrence of  $\xi$  in  $\alpha(\xi)$  by the term  $\tau'$ . As we said!

(b) Revisit Exercises 31, and render the informal arguments given there into suitable QL languages, and then provide formal derivations of the conclusions from the premisses. (Don't skip the propositional reasoning in these easy cases!)

(1) If Jo can do the exercises, then everyone in the class can do the exercises. Mo is in the class, and can't do the exercises. So Jo can't do the exercises.

Translation:  $(P \rightarrow \forall x(Cx \rightarrow Dx)), (Cm \wedge \neg Dm); \therefore \neg P$ .

For this example and the following ones, revisit the answers to Exercises 31 for informal motivation! Evidently our proof in this case is going to proceed by reductio, assuming P.

(1)	(P $\rightarrow$ $\forall x(Cx \rightarrow Dx)$ )	(Prem)																																	
(2)	(Cm $\wedge$ $\neg Dm$ )	(Prem)																																	
(3)	P	(Supp)																																	
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(2) *No whales are fish. So no fish are whales.*

We can translate ‘no’ propositions in two ways, using universal or existential quantification, giving us four alternative formal renditions of the argument. Let’s look at two:

$$\forall x(Wx \rightarrow \neg Fx) \therefore \forall x(Fx \rightarrow \neg Wx).$$

$$\neg \exists x(Wx \wedge Fx) \therefore \neg \exists x(Fx \wedge Wx).$$

For the first version we can argue in outline as follows:

(1)	$\forall x(Wx \rightarrow \neg Fx)$	(Prem)
(2)	$(Wa \rightarrow \neg Fa)$	( $\wedge E$ , 1)
(3)	$(Fa \rightarrow \neg Wa)$	(PL 2)
(4)	$\forall x(Fx \rightarrow \neg Wx)$	( $\forall I$ 3)

where at the third step we compress some reasoning with the connectives. The basic plan, then, is that we take an arbitrary representative object, and apply (1); we fiddle around to get another claim about the same arbitrary representative; and generalize to get (4). In detail:

(1)	$\forall x(Wx \rightarrow \neg Fx)$	(Prem)				
(2)	$(Wa \rightarrow \neg Fa)$	( $\wedge E$ , 1)				
(3)	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>Fa</math></td> <td style="padding-left: 5px;"></td> </tr> </table>	$Fa$		(Supp)		
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$\neg Wa$						
(8)	$(Fa \rightarrow \neg Wa)$	(CP 3–7)				
(9)	$\forall x(Fx \rightarrow \neg Wx)$	( $\forall I$ 8)				

For the second version, we obviously want a reductio argument of this shape:

	$\neg \exists x(Wx \wedge Fx)$	(Prem)						
	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <math>\exists x(Fx \wedge Wx)</math> </td> <td style="padding-left: 5px;"></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <math>\vdots</math> </td> <td style="padding-left: 5px;"></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <math>\perp</math> </td> <td style="padding-left: 5px;"></td> </tr> </table>	$\exists x(Fx \wedge Wx)$		$\vdots$		$\perp$		(Supp)
$\exists x(Fx \wedge Wx)$								
$\vdots$								
$\perp$								
	$\neg \exists x(Fx \wedge Wx)$	(RAA)						

So how are we going to fill in the dots? We will need to use the existentially quantified supposition: so evidently we will need to suppose an instance of it with an eye to invoking the rule ( $\exists E$ ). But with that hint, the proof more or less writes itself:

(1)	$\neg \exists x(Wx \wedge Fx)$	(Prem)				
(2)	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <math>\exists x(Fx \wedge Wx)</math> </td> <td style="padding-left: 5px;"></td> </tr> </table>	$\exists x(Fx \wedge Wx)$		(Supp)		
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$\perp$						
(10)	$\neg \exists x(Fx \wedge Wx)$	(RAA 2–9)				

(3) *All leptons have half-integer spin. All electrons are leptons. So all electrons have half-integer spin.*

Translation:  $\forall x(Lx \rightarrow Hx), \forall x(Ex \rightarrow Lx) \therefore \forall x(Ex \rightarrow Hx)$ .

We know we just need to think about an arbitrary item in the domain; to cheerfully muddle up our languages, the premisses tell us that if it is E it is L and hence H. So we can argue . . .

(1)	$\forall x(Lx \rightarrow Hx)$		(Prem)
(2)	$\forall x(Ex \rightarrow Lx)$		(Prem)
(3)	(La $\rightarrow$ Ha)		( $\forall E$ 1)
(4)	(Ea $\rightarrow$ La)		( $\forall E$ 2)
(5)	Ea		(Supp)
(6)	La		(MP 5, 4)
(7)	Ha		(MP 6, 3)
(8)	(Ea $\rightarrow$ Ha)		(CP 3–7)
(9)	$\forall x(Ex \rightarrow Hx)$		( $\forall I$ 8 – permissible since a is in no still-live assumption)

- (4) *Some chaotic attractors are not fractals. Every Cantor set is a fractal. Hence some chaotic attractors are not Cantor sets.*

Translation:  $\exists x(Ax \wedge \neg Fx), \forall x(Cx \rightarrow Fx) \therefore \exists x(Ax \wedge \neg Cx)$ .

The shape of the proof we need is plain:

	$\exists x(Ax \wedge \neg Fx)$		(Prem)
	$\forall x(Cx \rightarrow Fx)$		(Prem)
	(Aa $\wedge$ $\neg$ Fa)		(Supp)
	$\vdots$		
	$\exists x(Ax \wedge \neg Cx)$		
	$\exists x(Ax \wedge \neg Cx)$		( $\exists E$ )

That’s because we are evidently going to need to use the first, existentially quantified, premiss – and that will involve making a supposition as at the third line. And how can we derive the penultimate line? Presumably from an instance (Aa  $\wedge$   $\neg$ Ca):

	$\exists x(Ax \wedge \neg Fx)$		(Prem)
	$\forall x(Cx \rightarrow Fx)$		(Prem)
	(Aa $\wedge$ $\neg$ Fa)		(Supp)
	(Ca $\rightarrow$ Fa)		( $\forall E$ 2)
	$\vdots$		
	(Aa $\wedge$ $\neg$ Ca)		(by PL reasoning)
	$\exists x(Ax \wedge \neg Cx)$		( $\forall I$ )
	$\exists x(Ax \wedge \neg Cx)$		( $\exists E$ )

Filling in the dots, though, is easy propositional reasoning!

(1)	$\exists x(Ax \wedge \neg Fx)$		(Prem)
(2)	$\forall x(Cx \rightarrow Fx)$		(Prem)
(3)	(Aa $\wedge$ $\neg$ Fa)		(Supp)
(4)	(Ca $\rightarrow$ Fa)		( $\forall E$ 2)
(5)	Aa		( $\wedge E$ 3)
(6)	$\neg$ Fa		( $\wedge E$ 3)
(7)	Ca		(Supp)
(8)	Fa		(MP 7, 4)
(9)	$\perp$		(Abs 8, 6)
(10)	$\neg$ Ca		(RAA 7–9)
(11)	(Aa $\wedge$ $\neg$ Ca)		( $\wedge I$ 5, 10)
(12)	$\exists x(Ax \wedge \neg Cx)$		( $\forall I$ 11)
(13)	$\exists x(Ax \wedge \neg Cx)$		( $\exists E$ 1, 3–12)

- (5) *Some philosophers are logicians. All logicians are rational people. No rational person is a flat-earther. Therefore some philosophers are not flat-earthers.*

Translation:  $\exists x(Px \wedge Lx), \forall x(Lx \rightarrow Rx), \forall x(Rx \rightarrow \neg Fx) \therefore \exists x(Px \wedge \neg Fx)$ .

To make use of the first premiss, we'll want a proof of the shape

$\exists x(Px \wedge Lx)$	$(\text{Prem})$
$\forall x(Lx \rightarrow Rx)$	$(\text{Prem})$
$\forall x(Rx \rightarrow \neg Fx)$	$(\text{Prem})$
$(Pa \wedge La)$	$(\text{Supp})$
$\vdots$	
$\exists x(Px \wedge \neg Fx)$	
$\exists x(Px \wedge \neg Fx)$	$(\exists E \text{ from first premiss and the subproof})$

So filling in the details in the obvious way, we get

(1)	$\exists x(Px \wedge Lx)$	$(\text{Prem})$
(2)	$\forall x(Lx \rightarrow Rx)$	$(\text{Prem})$
(3)	$\forall x(Rx \rightarrow \neg Fx)$	$(\text{Prem})$
(4)	$(Pa \wedge La)$	$(\text{Supp})$
(5)	$(La \rightarrow Ra)$	$(\forall E 2)$
(6)	$(Ra \rightarrow \neg Fa)$	$(\forall E 3)$
(7)	$Pa$	$(\wedge E 4)$
(8)	$La$	$(\wedge E 4)$
(9)	$Ra$	$(MP 8, 5)$
(10)	$\neg Fa$	$(MP 9, 6)$
(11)	$(Pa \wedge \neg Fa)$	$(\wedge I 7, 10)$
(12)	$\exists x(Px \wedge \neg Fx)$	$(\exists I 11)$
(13)	$\exists x(Px \wedge \neg Fx)$	$(\exists E 1, 4-12)$

What if we'd used the other rendition of the 'no' premiss, so we translate the argument

Translation:  $\exists x(Px \wedge Lx), \forall x(Lx \rightarrow Rx), \neg \exists x(Rx \wedge Fx) \therefore \exists x(Px \wedge \neg Fx)$ ?

Then here's a derivation with the same basic shape, but with the details filled differently.

(1)	$\exists x(Px \wedge Lx)$	$(\text{Prem})$
(2)	$\forall x(Lx \rightarrow Rx)$	$(\text{Prem})$
(3)	$\neg \exists x(Rx \wedge Fx)$	$(\text{Prem})$
(4)	$(Pa \wedge La)$	$(\text{Supp})$
(5)	$(La \rightarrow Ra)$	$(\forall E 2)$
(6)	$Pa$	$(\wedge E 4)$
(7)	$La$	$(\wedge E 4)$
(8)	$Ra$	$(MP 7, 5)$
(9)	$Fa$	$(\text{Supp})$
(10)	$(Ra \wedge Fa)$	$(\wedge I 8, 9)$
(11)	$\exists x(Rx \wedge Fx)$	$(\exists I 10)$
(12)	$\perp$	$(\text{Abs } 11, 2)$
(13)	$\neg Fa$	$(\text{RAA } 9-12)$
(14)	$(Pa \wedge \neg Fa)$	$(\wedge I 7, 10)$
(15)	$\exists x(Px \wedge \neg Fx)$	$(\exists I 11)$
(16)	$\exists x(Px \wedge \neg Fx)$	$(\exists E 1, 4-12)$

A general point arising. If you want to make use of a premiss, assumption, or interim conclusion which is a *negated* existential proposition of the form  $\neg\exists\xi\alpha(\xi)$ , very often the thing to do is aim to prove the unnegated  $\exists\xi\alpha(\xi)$  from some assumption  $\beta$ . And then you'll have – as here – a reductio proof of  $\neg\beta$ .

- (6) *All lions and tigers are dangerous animals. Dangerous animals should be avoided. Leo is a lion. So Leo should be avoided.*

We indicated three possible regimentations of the first premiss in the answers to Exercises 31; let's formalize two of them to give the following two arguments:

$$\forall x((Lx \rightarrow Dx) \wedge (Tx \rightarrow Dx)), \forall x(Dx \rightarrow Ax) \text{ Lm} \therefore \text{Am}$$

$$\forall x((Lx \vee Tx) \rightarrow Dx), \forall x(Dx \rightarrow Ax) \text{ Lm} \therefore \text{Am}$$

The proofs are easy, either way:

(1)	$\forall x((Lx \rightarrow Dx) \wedge (Tx \rightarrow Dx))$	(Prem)
(2)	$\forall x(Dx \rightarrow Ax)$	(Prem)
(3)	Lm	(Prem)
(4)	$((Lm \rightarrow Dm) \wedge (Tm \rightarrow Dm))$	( $\forall E$ 1)
(5)	$(Dm \rightarrow Am)$	( $\forall E$ 2)
(6)	$(Lm \rightarrow Dm)$	( $\wedge E$ 4)
(7)	Dm	(MP 3, 6)
(8)	Am	(MP 7, 5)

(1)	$\forall x((Lx \vee Tx) \rightarrow Dx)$	(Prem)
(2)	$\forall x(Dx \rightarrow Ax)$	(Prem)
(3)	Lm	(Prem)
(4)	$((Lm \vee Tm) \rightarrow Dm)$	( $\forall E$ 1)
(5)	$(Dm \rightarrow Am)$	( $\forall E$ 2)
(6)	$(Lm \vee Tm)$	( $\vee I$ 3)
(7)	Dm	(MP 6, 4)
(8)	Am	(MP 7, 5)

- (c) *Also translate the following into suitable QL languages, and again provide formal derivations of the conclusions from the premisses:*

- (1) *If Jones is a bad philosopher, then some Welsh speaker is irrational; but every Welsh speaker is rational; hence Jones is not a bad philosopher.*

Translation:  $(P \rightarrow \exists x(Wx \wedge \neg Rx)), \forall x(Wx \rightarrow Rx) \therefore \neg P$

We are not interested here in the internal structure of the proposition 'Jones is a bad philosopher' so we can use a single letter (a zero-place predicate!) to represent it.

Since we want to prove  $\neg P$ , the obvious strategy for a formal proof is to assume 'P' and aim for a contradiction. Let's do it! So the proof will have this shape:

	$(P \rightarrow \exists x(Wx \wedge \neg Rx))$	(Prem)
	$\forall x(Wx \rightarrow Rx)$	(Prem)
	P	(Supp)
	$\exists x(Wx \wedge \neg Rx)$	(MP)
	⋮	
	⊥	
$\neg P$		(RAA)

How can we use the existential quantification at the fourth line? We obviously need to make a new supposition, supposing an instance [using a dummy name] is true, with a view to an argument by  $(\forall E)$ . Which suggests the argument, filled out a little more, will look like this:

(P $\rightarrow$ $\exists x(Wx \wedge \neg Rx)$ )	(Prem)
$\forall x(Wx \rightarrow Rx)$	(Prem)
P	(Supp)
$\exists x(Wx \wedge \neg Rx)$	(MP)
(Wa $\wedge$ Ra)	(Supp)
$\vdots$	
$\perp$	
$\perp$	( $\forall E$ from $\exists x(Wx \wedge \neg Rx)$ and subproof)
$\neg P$	(RAA)

But now the proof is quickly finished. We obviously need to use the second premiss, so instantiate that using the term ‘a’, and our desired contradiction is almost immediate:

1	(P $\rightarrow$ $\exists x(Wx \wedge \neg Rx)$ )	(Prem)
2	$\forall x(Wx \rightarrow Rx)$	(Prem)
3	P	(Supp)
4	$\exists x(Wx \wedge \neg Rx)$	(MP 3, 1)
5	(Wa $\wedge$ $\neg Ra$ )	(Supp)
6	Wa	( $\wedge E$ 5)
7	$\neg Ra$	( $\wedge E$ 5)
8	(Wa $\rightarrow$ Ra)	( $\forall E$ 2)
9	Ra	(MP 6, 8)
10	$\perp$	(Abs 9, 7)
11	$\perp$	( $\exists E$ 4, 5–10)
12	$\neg P$	(RAA)

- (2) *Everyone is such that, if they admire Ludwig, then the world has gone mad. Therefore, if someone admires Ludwig, the world has gone mad.*

Translation:  $\forall x(Lx \rightarrow P) \therefore (\exists x Lx \rightarrow P)$

Here, the quantifier runs over the relevant people. We need expose no internal structure in the predicate ‘admires Ludwig’, so a simple unary predicate will suffice. And again we are not interested here in the internal structure of the proposition ‘the world has gone mad’ so we can use a single letter to represent it.

Here’s overall proof shape we will evidently need (what else can we do but assume the conclusion is false and aim for a reductio?)

$\forall x(Lx \rightarrow P)$	(Prem)
$\neg(\exists x Lx \rightarrow P)$	(Supp)
$\vdots$	
$\perp$	
$\neg\neg(\exists x Lx \rightarrow P)$	(RAA)
$(\exists x Lx \rightarrow P)$	(DN)

Now, let’s allow ourselves to skip the detail of PL reasoning; we know that from a wff of the form  $\neg(\alpha \rightarrow \beta)$  we can derive both  $\alpha$  and  $\neg\beta$ . So apply that to our supposition, and we get:

$\forall x(Lx \rightarrow P)$	(Prem)
$\neg(\exists x Lx \rightarrow P)$	(Supp)
$\exists x Lx$	(by PL reasoning from the supposition)
$\neg P$	(by PL reasoning from the supposition)
$\vdots$	
$\perp$	
$\neg\neg(\exists x Lx \rightarrow P)$	(RAA)
$(\exists x Lx \rightarrow P)$	(DN)

So now how can we use the existential quantification on the third line? We will need to set an inference by  $(\exists E)$  by taking an instance of it as a further supposition. So we will now expect to get

$\forall x(Lx \rightarrow P)$	(Prem)
$\neg(\exists x Lx \rightarrow P)$	(Supp)
$\exists x Lx$	(by PL reasoning from the supposition)
$\neg P$	(by PL reasoning from the supposition)
$\vdots$	
$L_a$	(Supp)
$\vdots$	
$\perp$	
$\perp$	( $\exists E$ using wff at the third line and the subproof)
$\neg\neg(\exists x Lx \rightarrow P)$	(RAA)
$(\exists x Lx \rightarrow P)$	(DN)

And now linking the new supposition to the initial premiss, the proof finishes itself very easily!

(1)	$\forall x(Lx \rightarrow P)$	(Prem)
(2)	$\neg(\exists x Lx \rightarrow P)$	(Supp)
(3)	$\exists x Lx$	(PL 2)
(4)	$\neg P$	(PL 2)
(5)	$L_a$	(Supp)
(6)	$(L_a \rightarrow P)$	( $\forall E$ 1)
(7)	$P$	(MP 5, 6)
(8)	$\perp$	(Abs 7, 4)
(9)	$\perp$	( $\exists E$ 3, 4–8)
(10)	$\neg\neg(\exists x Lx \rightarrow P)$	(RAA 2–9)
(11)	$(\exists x Lx \rightarrow P)$	(DN 10)

- (3) *Jack is taller than Jill. Someone is taller than Jack. If a first person is taller than a second, and the second is taller than a third, then the first person is taller than the third. Hence someone is taller than both Jack and Jill.*

Translation:  $Tmn, \exists xTxm, \forall x\forall y\forall z((Txy \wedge Tyz) \rightarrow Txz) \therefore \exists x(Lxm \wedge Txn)$

Arguing informally, the second premiss tells us that there is someone (Alice, if you like) who is taller than Jack. So Alice is taller than Jack who is taller than Jill. By the third premiss, it follows that Alice is taller than Jill. So indeed there is someone (Alice!) who is taller than both Jack and Jill.

Now we turn that informal line of argument into a formal deduction. Do note, by the way, that formally our  $(\forall I)$  only allows us to *instantiate one quantifier at a time* from a wff like the second premiss.

(1)	Tmn		(Prem)				
(2)	$\exists xTxm$		(Prem)				
(3)	$\forall x\forall y\forall z((Txy \wedge Tyz) \rightarrow Txz)$		(Prem)				
(4)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">Tam</td> <td style="width: 10%;"></td> <td style="width: 70%;">(Supp)</td> </tr> </table>		Tam		(Supp)		
	Tam		(Supp)				
(5)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">(Tam <math>\wedge</math> Tmn)</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\wedge</math>I 4, 3)</td> </tr> </table>		(Tam $\wedge$ Tmn)		( $\wedge$ I 4, 3)		
	(Tam $\wedge$ Tmn)		( $\wedge$ I 4, 3)				
(6)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\forall y\forall z((Tay \wedge Tyz) \rightarrow Taz)</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\forall</math>E 3)</td> </tr> </table>		$\forall y\forall z((Tay \wedge Tyz) \rightarrow Taz)$		( $\forall$ E 3)		
	$\forall y\forall z((Tay \wedge Tyz) \rightarrow Taz)$		( $\forall$ E 3)				
(7)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\forall z((Tam \wedge Tmz) \rightarrow Taz)</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\forall</math>E 5)</td> </tr> </table>		$\forall z((Tam \wedge Tmz) \rightarrow Taz)$		( $\forall$ E 5)		
	$\forall z((Tam \wedge Tmz) \rightarrow Taz)$		( $\forall$ E 5)				
(8)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">((Tam <math>\wedge</math> Tmn) <math>\rightarrow</math> Tan)</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\forall</math>E 5)</td> </tr> </table>		((Tam $\wedge$ Tmn) $\rightarrow$ Tan)		( $\forall$ E 5)		
	((Tam $\wedge$ Tmn) $\rightarrow$ Tan)		( $\forall$ E 5)				
(9)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">Tan</td> <td style="width: 10%;"></td> <td style="width: 70%;">(MP 5, 8)</td> </tr> </table>		Tan		(MP 5, 8)		
	Tan		(MP 5, 8)				
(10)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">(Tam <math>\wedge</math> Tan)</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\wedge</math>E 10)</td> </tr> </table>		(Tam $\wedge$ Tan)		( $\wedge$ E 10)		
	(Tam $\wedge$ Tan)		( $\wedge$ E 10)				
(11)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\exists x(Lxm \wedge Txn)</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\exists</math>I 9)</td> </tr> </table>		$\exists x(Lxm \wedge Txn)$		( $\exists$ I 9)		
	$\exists x(Lxm \wedge Txn)$		( $\exists$ I 9)				
(12)	$\exists x(Lxm \wedge Txn)$		( $\exists$ E 3, 4–10)				

- (4) *Every logician admires Gödel. Whoever admires someone is not without feeling. Hence no logician is without feeling.*

Using an obvious glossary, we have the following possible translations:

$$\forall x(Lx \rightarrow A_{xg}), \forall x(\exists yA_{xy} \rightarrow \neg Wx) \therefore \forall x(Lx \rightarrow \neg Wx)$$

$$\forall x(Lx \rightarrow A_{xg}), \forall x(\exists yA_{xy} \rightarrow \neg Wx) \therefore \neg \exists x(Lx \wedge Wx)$$

In the first case, we need to prove a universal generalization. Aim to establish an arbitrary instance of it,  $(La \rightarrow \neg Wa)$ . That's a conditional, so suppose  $La$  and aim for  $\neg Wa \dots$

(1)	$\forall x(Lx \rightarrow A_{xg})$		(Prem)				
(2)	$\forall x(\exists yA_{xy} \rightarrow \neg Wx)$		(Prem)				
(3)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">La</td> <td style="width: 10%;"></td> <td style="width: 70%;">(Supp)</td> </tr> </table>		La		(Supp)		
	La		(Supp)				
(4)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">(La <math>\rightarrow</math> Aag)</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\forall</math>E 1)</td> </tr> </table>		(La $\rightarrow$ Aag)		( $\forall$ E 1)		
	(La $\rightarrow$ Aag)		( $\forall$ E 1)				
(5)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">Aag</td> <td style="width: 10%;"></td> <td style="width: 70%;">(MP 3, 4)</td> </tr> </table>		Aag		(MP 3, 4)		
	Aag		(MP 3, 4)				
(6)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">(<math>\exists yA_{ay} \rightarrow \neg Wa</math>)</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\forall</math>E 2)</td> </tr> </table>		( $\exists yA_{ay} \rightarrow \neg Wa$ )		( $\forall$ E 2)		
	( $\exists yA_{ay} \rightarrow \neg Wa$ )		( $\forall$ E 2)				
(7)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\exists yA_{ay}</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\exists</math>I 5)</td> </tr> </table>		$\exists yA_{ay}$		( $\exists$ I 5)		
	$\exists yA_{ay}$		( $\exists$ I 5)				
(8)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\neg Wa</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(MP 7, 6)</td> </tr> </table>		$\neg Wa$		(MP 7, 6)		
	$\neg Wa$		(MP 7, 6)				
(9)	(La $\rightarrow$ $\neg Wa$ )		(CP 3–8)				
(10)	$\forall x(Lx \rightarrow \neg Wx)$		( $\forall$ I 9)				

For the second version, we argue

(1)	$\forall x(Lx \rightarrow A_{xg})$		(Prem)								
(2)	$\forall x(\exists yA_{xy} \rightarrow \neg Wx)$		(Prem)								
(3)	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"><math>\exists x(Lx \wedge Wx)</math></td> <td style="width: 10%;"></td> <td style="width: 70%;">(Supp)</td> </tr> </table>		$\exists x(Lx \wedge Wx)$		(Supp)						
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	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%;"></td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;">Wa</td> <td style="width: 10%;"></td> <td style="width: 70%;">(<math>\wedge</math>E 4)</td> </tr> </table>		Wa		( $\wedge$ E 4)		( $\wedge$ E 4)				
	Wa		( $\wedge$ E 4)								
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	(La $\rightarrow$ Aag)		( $\forall$ E 1)								
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	$\perp$		(Abs 6, 11)								
(13)	$\perp$		( $\exists$ E 3, 4–12)								
(14)	$\neg \exists x(Lx \wedge Wx)$		(RAA 3–13)								



- (5) *Either not everyone liked the cake or someone baked an excellent cake. If I'm right, then whoever bakes an excellent cake ought to be proud. So if everyone liked the cake and I'm right, then someone ought to be proud.*

Note that the internal structure of 'everyone liked the cake' does no work at all, so we might as well represent it as an unstructured proposition  $E$ . Again the internal structure of the predicate 'baked an excellent cake' does no work: so we might as well represent it with an unstructured unary predicate  $B$ . Which gives us the rendition

Translation:  $(\neg E \vee \exists yBy)$ ,  $(R \rightarrow \forall x(Bx \rightarrow Px)) \therefore ((E \wedge R) \rightarrow \exists xPx)$

To prove the conditional conclusion, assume the antecedent. Using disjunctive syllogism gives us  $\exists yBy$  from the first premiss. Using modus ponens gives us  $\forall x(Bx \rightarrow Px)$  from the second premiss. But those two quantified propositions yield  $\exists xPx$  in a now familiar way. So ....

(1)	$(\neg E \vee \exists yBy)$		(Prem)
(2)	$(R \rightarrow \forall x(Bx \rightarrow Px))$		(Prem)
(3)	$(E \wedge R)$		(Supp)
(4)	$E$		( $\wedge E$ 3)
(5)	$R$		( $\wedge E$ 3)
(6)	$\neg E$		(Supp)
(7)	$\perp$		(Abs 4, 6)
(8)	$\exists yBy$		(EFQ 7)
(9)	$\exists yBy$		(Supp)
(10)	$\exists yBy$		(Iter 9)
(11)	$\exists yBy$		( $\vee E$ 1, 6–8, 9–10)
(12)	$\forall x(Bx \rightarrow Px)$		(MP 4, 2)
(13)	$Ba$		(Supp)
(14)	$(Ba \rightarrow Pa)$		( $\forall E$ 12)
(15)	$Pa$		(MP 13, 14)
(16)	$\exists xPx$		( $\exists I$ 15)
(17)	$\exists xPx$		( $\exists E$ 11, 13–16)
(18)	$((E \wedge R) \rightarrow \exists xPx)$		(CP 3–17)

Also translate the following into suitable QL languages and show that they are logical truths by deriving them as theorems:

- (6) *Everyone is either a logician or not a logician.*

We want to prove something of the form  $\forall x(Lx \vee \neg Lx)$ . We just (i) prove  $(La \wedge \neg La)$  from no assumptions and then (ii) universally generalize. Step (i) is bookwork. Step (ii) is now trivial.

(1)			
(2)	$\neg(La \vee \neg La)$		(Supp)
(3)	$La$		(Supp)
(4)	$(La \vee \neg La)$		( $\vee I$ 3)
(5)	$\perp$		(Abs 4, 2)
(6)	$\neg La$		(RAA 3–5)
(7)	$(La \vee \neg La)$		( $\vee I$ 6)
(8)	$\perp$		(Abs 7, 2)
(9)	$\neg\neg(La \vee \neg La)$		(RAA 2–8)
(10)	$(La \vee \neg La)$		(DN 9)
(11)	$\forall x(Lx \vee \neg Lx)$		( $\forall I$ 10)



(1)			
(2)		$\neg\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lay))$	(Supp)
(3)		(Lab $\wedge$ Lbb)	(Supp)
(4)		$((Lab \wedge Lbb) \wedge (\neg\neg Lbb \rightarrow \neg Lab))$	( $\vee$ I 3)
(5)		$\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lay))$	( $\exists$ I 4)
(6)		$\perp$	(5, 2)
(7)		$\neg(Lab \wedge Lbb)$	(RAA 3–6)
(8)		$(\neg\neg Lbb \rightarrow \neg Lab)$	(PL 7)
(9)		$((Lab \wedge Lbb) \wedge (\neg\neg Lbb \rightarrow \neg Lab))$	( $\vee$ I 8)
(10)		$\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lay))$	( $\exists$ I 9)
(11)		$\perp$	(Abs 10, 2)
(12)		$\neg\neg\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lay))$	(RAA 10, 2)
(13)		$\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lay))$	(DN 11)
(14)		$\forall x\exists y((Lxy \wedge Lyy) \vee (\neg\neg Lyy \rightarrow \neg Lxy))$	( $\forall$ I 12)

You can be trusted by now to fill in the PL reasoning from (7) to (8)! And if you'd used the other, disjunctive, rendition of 'unless' to translate the argument, the derivation would have gone very similarly.

(d) Give proofs to warrant the following inferences (compare §29.6):

(1)  $(P \wedge \forall x Fx) \therefore \forall x(P \wedge Fx)$

This is very straightforward! We have to unpack the conjunctive premiss. We then need to aim to prove something like  $(P \wedge Fa)$  so we can universally quantify to get the desired conclusion. We've already got  $P$ . How can we get  $Fa$ ?

(1)	(P $\wedge$ $\forall x Fx$ )	(Prem)
(2)	P	( $\wedge$ E 1)
(3)	$\forall x Fx$	( $\wedge$ E 1)
(4)	Fa	( $\forall$ E 3)
(5)	(P $\wedge$ Fa)	( $\wedge$ I 2, 4)
(6)	$\forall x(P \wedge Fx)$	( $\forall$ I 5)

(2)  $(\exists y Fy \vee P) \therefore \exists y(Fy \vee P)$

We'll need to argue by cases. So, first case, we show  $\exists y Fy$  implies  $\exists y(Fy \vee P)$ . This will be a pretty simple ( $\exists$ E) argument.

Then, second case, we need to show  $P$  also implies  $\exists y(Fy \vee P)$ . But that only takes one intermediate step!

(1)	(1)	$(\exists y Fy \vee P)$	(Prem)
(2)		$\exists y Fy$	(Supp)
(3)		Fa	(Supp)
(4)		$(Fa \vee P)$	( $\vee$ I 3)
(5)		$\exists y(Fy \vee P)$	( $\exists$ I 4)
(6)		$\exists y(Fy \vee P)$	( $\exists$ E 2, 3–5)
(7)		P	(Supp)
(8)		$(Fa \vee P)$	( $\vee$ I 7)
(9)		$\exists y(Fy \vee P)$	( $\exists$ I 8)
(10)		$\exists y(Fy \vee P)$	( $\vee$ E 1, 2–6, 7–9)



(1)	$\forall x(Fx \vee \neg Fx)$	(Prem)
(2)	$(Fa \vee \neg Fa)$	( $\forall E$ 1)
(3)	$Fa$	(Supp)
(4)	$\forall xFx$	( $\forall I$ 3 [? after all, a was arbitrary!])
(5)	$(\forall xFx \vee \forall x\neg Fx)$	( $\forall I$ 4)
(6)	$\neg Fx$	(Supp)
(7)	$\forall x\neg Fx$	( $\forall I$ 6 [? after all, a was arbitrary!])
(8)	$(\forall xFx \vee \forall x\neg Fx)$	( $\forall I$ 7)
(9)	$(\forall xFx \vee \forall x\neg Fx)$	( $\forall E$ 2, 3-5, 6-8)

This so-called proof is a disaster. The culprit is our double offence against the constraints on using ( $\forall I$ ), flouting the rules at lines (4) and (7).

So that leaves us with the question, how *can* we warrant the evidently correct inference  $\forall x(P \vee Fx) \therefore (P \vee \forall x Fx)$  using our rules?

Well, start the proof in the same way, but then use that familiar dodge of assuming the negation of the desired conclusion and aiming for a reductio:

(1)	$\forall x(P \vee Fx)$	(Prem)
(2)	$(P \vee Fa)$	( $\forall E$ 1)
(3)	$\neg(P \vee \forall x Fx)$	(Supp)
(4)	$P$	(Supp)
(5)	$(P \vee \forall x Fx)$	( $\forall I$ 4)
(6)	$\perp$	(Abs 5, 3)
(7)	$\neg P$	(RAA 4-6]
(8)	$Fa$	(PL – disjunctive syllogism from 7 and 2)
(9)	$\forall xFx$	( $\forall I$ 8, legitimate because a is in no assumption)
(10)	$(P \vee \forall x Fx)$	( $\forall I$ 9)
(11)	$\perp$	(Abs 10, 3)
(12)	$\neg\neg(P \vee \forall x Fx)$	(RAA 3-11)
(13)	$(P \vee \forall x Fx)$	(DN 12)

So, to make this into a proof by our basic rules, we'd just have to use the usual routine to flesh out the disjunctive syllogism move. I'll leave that to you!

(5)  $(P \rightarrow \exists x Fx) \therefore \exists x(P \rightarrow Fx)$

This fourth example is trickier. One option would be to invoke  $(P \vee \neg P)$  and then argue by cases. First case,  $P$  and the premiss gives  $\exists x Fx$  and then we can argue to the desired conclusion using an ( $\exists E$ ) argument when we remember that e.g.  $Fa$  implies  $(P \rightarrow Fa)$  by propositional reasoning. Second case,  $\neg P$  implies  $(P \rightarrow Fa)$  by propositional reasoning and again we get our conclusion.

But that's not very pretty! Let's see if we can give an alternative derivation using just the negation and conditional rules. OK: what can now we do but take the premiss and suppose the negation of the conclusion and aim for absurdity. How can we use the conditional? Well we can suppose in addition that the antecedent is true and derive the consequent:

	$(P \rightarrow \exists x Fx)$	(Prem)
	$\neg\exists x(P \rightarrow Fx)$	(Supp)
	$P$	(Supp)
	$\exists x Fx$	(MP)
	$\vdots$	

But now what? We have no useful move to make with the wffs already in play (why not?). So we are going to need to make *another* supposition. And the obvious one to make is  $Fa$  [of course, the particular choice of dummy name doesn't matter!] with a view to setting up a  $(\exists E)$  argument using the existential wff on the fourth line .....

$(P \rightarrow \exists x Fx)$	(Prem)
$\neg \exists x(P \rightarrow Fx)$	(Supp)
$P$	(Supp)
$\exists x Fx$	(MP)
$Fa$	(Supp)
$\vdots$	

Now what to do? Let's look for a contradiction! Well, by propositional reasoning  $Fa$  gives us  $(P \rightarrow Fa)$  – a [material] conditional is true if its consequent is true – and if we existentially quantify that we can have a contradiction with the first of our suppositions. Hooray!

$(P \rightarrow \exists x Fx)$	(Prem)
$\neg \exists x(P \rightarrow Fx)$	(Supp)
$P$	(Supp)
$\exists x Fx$	(MP)
$Fa$	(Supp)
$P$	(Supp)
$Fa$	(Iter)
$(P \rightarrow Fa)$	(CP)
$\exists x(P \rightarrow Fx)$	( $\exists I$ )
$\perp$	(Abs)
$\perp$	( $\exists E$ from fourth line and the ensuing subproof)
$\vdots$	

So now where? We could use reductio to derive  $\neg P$ ; but how would that help? By propositional reasoning we can derive  $(P \rightarrow Fa)$  again, so  $\exists x(P \rightarrow Fx)$ , and so contradiction again. Good! But we can be brisker:

(1)	$(P \rightarrow \exists x Fx)$	(Prem)
(2)	$\neg \exists x(P \rightarrow Fx)$	(Supp)
(3)	$P$	(Supp)
(4)	$\exists x Fx$	(MP 3, 1)
(5)	$Fa$	(Supp)
(6)	$P$	(Supp)
(7)	$Fa$	(Iter 5)
(8)	$(P \rightarrow Fa)$	(CP 6–7)
(9)	$\exists x(P \rightarrow Fx)$	( $\exists I$ 8)
(10)	$\perp$	(Abs 9, 2)
(11)	$\perp$	( $\exists E$ 4, 5–10)
(12)	$Fa$	(EFQ 11)
(13)	$(P \rightarrow Fa)$	(CP 3–12)
(14)	$\exists x(P \rightarrow Fx)$	( $\exists I$ 13)
(15)	$\perp$	(Abs 14, 2)
(16)	$\neg \neg \exists x(P \rightarrow Fx)$	(RAA 2–15)
(17)	$\exists x(P \rightarrow Fx)$	(DN 16)

---

(e\*) Suppose we set up QL-style languages with only the universal quantifier added to the connectives. Expressions of the form  $\exists\xi\alpha(\xi)$  are now introduced into such a language simply as abbreviations for corresponding expressions of the form  $\neg\forall\xi\neg\alpha(\xi)$ . Show that the two familiar existential quantifier rules would then be derived rules of this system.

We need to show that

1. We can preserve  $(\exists I)$  in the form of the rule “We can derive the wff  $\neg\forall\xi\neg\alpha(\xi)$  [ $\exists\xi\alpha(\xi)$  for short] from any given instance  $\alpha(\tau)$ .”
2. We can preserve  $(\exists E)$  in the form of the rule “Given  $\neg\forall\xi\neg\alpha(\xi)$  [ $\exists\xi\alpha(\xi)$  for short], and a subproof from the instance  $\alpha(\delta)$  as supposition to the conclusion  $\gamma$  – where (i) the dummy name  $\delta$  is new to the proof, and (ii)  $\gamma$  does not contain  $\delta$  – then we can derive  $\gamma$ .”

Then for (1) we just note that we can always argue in a pattern like this:

$\alpha(\tau)$							
$\vdots$							
<table style="border-collapse: collapse; margin-left: 20px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\forall\xi\neg\alpha(\xi)</math></td> <td>(Supp)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\neg\alpha(\tau)</math></td> <td>(<math>\forall E</math>)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\perp</math></td> <td>(Abs)</td> </tr> </table>	$\forall\xi\neg\alpha(\xi)$	(Supp)	$\neg\alpha(\tau)$	( $\forall E$ )	$\perp$	(Abs)	
$\forall\xi\neg\alpha(\xi)$	(Supp)						
$\neg\alpha(\tau)$	( $\forall E$ )						
$\perp$	(Abs)						
$\neg\forall\xi\neg\alpha(\xi)$	(RAA)						

For (2) note that we can always argue in a pattern like this:

$\neg\forall\xi\neg\alpha(\xi)$															
<table style="border-collapse: collapse; margin-left: 20px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\neg\gamma</math></td> <td>(Supp)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <table style="border-collapse: collapse; margin-left: 20px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\alpha(\delta)</math></td> <td>(Supp with <math>\delta</math> a new name)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\vdots</math></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\gamma</math></td> <td>(We are assuming there is such a proof from <math>\alpha(\delta)</math> to <math>\gamma</math>)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\perp</math></td> <td>(Abs)</td> </tr> </table> </td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\neg\alpha(\delta)</math></td> <td>(RAA)</td> </tr> </table>	$\neg\gamma$	(Supp)	<table style="border-collapse: collapse; margin-left: 20px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\alpha(\delta)</math></td> <td>(Supp with <math>\delta</math> a new name)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\vdots</math></td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\gamma</math></td> <td>(We are assuming there is such a proof from <math>\alpha(\delta)</math> to <math>\gamma</math>)</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>\perp</math></td> <td>(Abs)</td> </tr> </table>	$\alpha(\delta)$	(Supp with $\delta$ a new name)	$\vdots$		$\gamma$	(We are assuming there is such a proof from $\alpha(\delta)$ to $\gamma$ )	$\perp$	(Abs)		$\neg\alpha(\delta)$	(RAA)	
$\neg\gamma$	(Supp)														
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$\perp$	(Abs)														
$\neg\alpha(\delta)$	(RAA)														
$\forall\xi\neg\alpha(\xi)$	( $\forall I$ , legitimate because $\delta$ is in no undischarged assumption)														
$\perp$	(Abs)														
$\neg\neg\gamma$	(RAA)														
$\gamma$	(DN)														

Alternatively, suppose we set up a QL-style language with only a ‘no’ quantifier, expressed using the quantifier-former ‘N’ (so  $N\xi\alpha(\xi)$  holds when nothing satisfies the condition expressed by  $\alpha$ ). Give introduction and elimination rules for this quantifier. Define the universal and existential quantifier in this new language, and show how to recover their usual inference rules.

$N\xi\alpha(\xi)$  is equivalent to  $\forall\xi\neg\alpha(\xi)$ . So that suggests the obvious rules for ‘N’:

1. (NE) From  $N\xi\alpha(\xi)$ , we can infer  $\neg\alpha(\tau)$  for any term  $\tau$ .
2. (NI) From  $\neg\alpha(\delta)$  infer  $N\xi\alpha(\xi)$ , where  $\delta$  is a dummy name not in any live assumption for  $\neg\alpha(\delta)$ , and  $\delta$  doesn’t occur in  $N\xi\alpha(\xi)$ .

Obviously we can define  $\forall$  and  $\exists$  in terms of N. Thus:

- $\forall\xi\alpha(\xi)$  can be defined as abbreviating  $N\xi\neg\alpha(\xi)$
- $\exists\xi\alpha(\xi)$  can be defined as abbreviating  $\neg N\xi\alpha(\xi)$

And it is then easy to check that we can then recover the familiar inference rules. Thus we have

$\mathbf{N}\xi\neg\alpha(\xi)$	$[= \forall\xi\alpha(\xi)]$	
$\neg\neg\alpha(\tau)$		(NE, with $\neg\alpha$ for $\alpha$ )
$\alpha(\tau)$		(DN)

and

$\alpha(\delta)$		( $\delta$ not in any live assumption for this wff)
$\neg\neg\alpha(\delta)$		(PL reasoning)
$\mathbf{N}\xi\neg\alpha(\xi)$	$[= \forall\xi\alpha(\xi)]$	(NI, $\delta$ is not in this wff)

Similarly we get

$\alpha(\tau)$											
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$\neg\mathbf{N}\xi\alpha(\xi)$	$[= \exists\xi\alpha(\xi)]$										

And further, following the answer above to the first half of this question,

$\neg\mathbf{N}\xi\alpha(\xi)$	$[= \exists\xi\alpha(\xi)]$																									
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(f\*) *When discussing pairs of PL introduction and elimination rules, we saw that they fitted together in a harmonious way, with the elimination rule as it were reversing or undoing an application of the introduction rule. Can something similar be said about the pairs of QL introduction and elimination rules?*

Our earlier discussion of this idea of harmony between introduction and elimination rules was rather loose. The idea goes back to Gerhard Gentzen, the prime originator of natural deduction methods, who wrote

The introductions represent, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol [i.e. main operator] we are dealing only ‘in the sense afforded it by the introduction of that symbol’.

It isn’t easy to give a precise elaboration of this idea, however. So we will continue to proceed in a rather loose, arm-waving way!

OK: The rule ( $\forall$ I) – in its official form – tells us that given the instance  $\alpha(\delta)$  we can infer the universal quantification  $\forall\xi\alpha(\xi)$  (so long as the dummy name  $\delta$  doesn’t appear in any live assumption on which  $\alpha(\delta)$  depends).



Note, though, that the distinction between dummy names and proper names – while a very helpful device – isn't really of the essence here. What matters is that we are taking a representative object from the domain – i.e. an object about which we are making no special assumptions – and then if we can establish that this object satisfies some condition  $C$ , we can conclude that everything satisfies  $C$ . It doesn't really matter what kind of term we use in designating the representative object and arguing about it.

So really our rule might as well have been (in many natural deduction systems it *is*) this: given the instance  $\alpha(\tau)$ , we can infer the universal quantification  $\forall\xi\alpha(\xi)$  so long as  $\tau$  doesn't appear in any live assumption.

Hence: we can think of a universal quantification  $\forall\xi\alpha(\xi)$  as standardly arising by generalizing on any instance involving a term (so long as we make use of no special, distinguishing, info about that instance). Therefore arguing backwards from  $\forall\xi\alpha(\xi)$  to recover its standard grounds, that will take us back from the quantified wff to any instance. Which gives what the rule ( $\forall E$ ) says.

The rule ( $\exists I$ ) takes us from a particular instance  $\alpha(\tau)$  to the the existential quantification  $\exists\xi\alpha(\xi)$ . Now, that inference destroys information about which instance we start from, so we can't recover its particular grounds from a wff like  $\exists\xi\alpha(\xi)$ . However, suppose we have a general recipe taking us from an arbitrary satisfier of the condition expressed by  $\alpha$  to some fixed conclusion  $\gamma$ . Then given  $\exists\xi\alpha(\xi)$ , we know it standardly comes from *some* instance, so we know we could have applied our general recipe to conclude  $\gamma$ . Which is in effect what ( $\exists E$ ) says: given  $\exists\xi\alpha(\xi)$  and a general recipe for getting from  $\alpha(\delta)$  to the fixed conclusion  $\gamma$  [general in that we make no assumptions about  $\delta$ , a fixed conclusion that doesn't depend on  $\delta$ ] we can infer  $\gamma$ .