Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 17: The absurdity sign

(1) *Show that for any wff* α, $(\alpha \land \bot) \approx \bot$ *and* $(\alpha \land \neg\bot) \approx \alpha$. *What are the analogous results for wffs of the form* $(\alpha \lor \bot)$ *and* $(\alpha \lor \neg\bot)$*?*

This is almost trivial. A conjunction with a false conjunct is false; so since $\bot$ is always false, $(\alpha \land \bot)$ is always false.

Now, what does a metalinguistic claim of the form $\beta \approx \gamma$ say? That when we take any valuation of the atoms in β and γ, those two wffs have the same value. So what does $(\alpha \land \bot) \approx \bot$ say? On any valuation of the (ordinary) atoms in α [there are none in $\bot$] $(\alpha \land \bot)$ has the same value as $\bot$, i.e. is false. Which we've just seen is true.

Similarly, the value of a conjunction with a true conjunct is true or false depending on the value of the other conjunct. Since $\neg\bot$ is always true, $(\alpha \land \neg\bot)$ will always have the value of α. Hence $(\alpha \land \neg\bot) \approx \alpha$.

When we remember (i) that the value of disjunction with a false disjunct will be the value of the other disjunct, (ii) the value of a disjunction with a true disjunct is true (irrespective of the value of the other disjunct),and $\neg\bot$ is true on any valuation, it immediately follows that $(\alpha \lor \bot) \approx \alpha$ and $(\alpha \lor \neg\bot) \approx \neg\bot$.

(2) *Show that any wff including one or more occurrences of* $\bot$ *is equivalent to a wff without any absurdity signs or to* $\bot$ *or to* $\neg\bot$.

Here's the basic thought. Take a wff built up using $\neg, \land, \lor$ and $\bot$. We can ignore any adjacent pairs of negation signs and still have an equivalent wff. Then work on a wff from the inside out. We can replace a conjunction/disjunction of a wff α with $\bot$ or $\neg\bot$ with (as we have seen) with a simpler equivalent, either the wff α, $\bot$ or $\neg\bot$ as the case may be. Keep on going simplifying away (removing any new double negations that appear), and we'll be left at the end be left with either a wff without any remaining occurrences of $\bot$ or with just $\bot$ or just $\neg\bot$. For simple examples, noting that $(\mathsf{P} \land \bot) \approx \bot$, consider

$$(\neg(\mathsf{P} \land \bot) \lor ((\mathsf{Q} \land \mathsf{R}) \lor \bot)) \approx (\neg\bot \lor ((\mathsf{Q} \land \mathsf{R}) \lor \bot)) \approx ((\mathsf{Q} \land \mathsf{R}) \lor \bot) \approx (\mathsf{Q} \land \mathsf{R})$$

or noting that $(\bot \land \neg(\neg\mathsf{P} \lor \mathsf{R}) \approx \bot$ [the order of conjuncts doesn't matter! – if one conjunct is $\bot$ the whole wff is equivalent to $\bot$],

$$\neg(\neg(\bot \land \neg(\neg\mathsf{P} \lor \mathsf{R})) \land (\neg\bot \lor \mathsf{Q})) \approx \neg(\neg\bot \land (\neg\bot \lor \mathsf{Q}) \approx \neg(\neg\bot \lor \mathsf{Q}) \approx \neg\neg\bot \approx \bot$$

Similarly for more complex cases. [You can make this story more rigorous by considering replacement operations on parse trees – but doing this doesn't produce any more enlightenment!]

(3) *Find a binary connective % such that a language with just that connective is* not *expressively complete, but a language with % plus* $\bot$ *is expressively complete.*

The uparrow and downarrow connectives which we met at the end of Chapter 13 are expressively complete as they stand. But now consider these further two arrow connectives:

α	β	$(\alpha \to \beta)$	$(\alpha \leftarrow \beta)$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Unlike the up and down arrows, these right and left arrows are not expressively complete in themselves (you can't define negation from them – why?). However:

$(\alpha \to \bot)$ is equivalent to $\neg\alpha$, and – noting that the truth table for $\to$ makes $(\alpha \to \beta)$ equivalent to $(\neg\alpha \vee \beta)$ – we quickly see that $((\alpha \to \bot) \to \beta)$ is equivalent to $(\alpha \vee \beta)$. Since we can define every truth-function using $\neg$ and $\vee$, and we *can* define $\neg$ and $\vee$ from $\to$ and $\bot$, it follows that $\to$ and $\bot$ are expressively complete.

Similarly $(\bot \leftarrow \beta)$ is equivalent to $\neg\beta$ etc., and $\leftarrow$ and $\bot$ are expressively complete too.

How can we find these two candidates for %? Well, note (i) if $(\alpha \,\%\, \beta)$ is F when α and β are F, then any wff constructed from a false atom and the always-false $\bot$ using % will be false. So we couldn't construct negation which flips values. So the last line of a truth-table for % has to have the entry T. You can then consider the remaining cases, and very quickly rule out six.

Recalling the notation introduced in Exercises 16(c), where 'Γ' stands in for some wffs – zero, one, or many:*

(4*) *Show that* $\Gamma \vDash \bot$ *if and only if the wffs* Γ *are tautologically inconsistent.*

This just falls out of the definitions. To belabour the reasoning, we know by definition of the turnstile,

(i) $\Gamma \vDash \bot$

if and only if

(ii) Any valuation of the relevant atoms which makes Γ all true together makes $\bot$ true.

But *no* valuation ever makes $\bot$ true. So (ii) can only be true if and only if

(iii) No valuation of the relevant atoms makes Γ all true together.

But by definition (iii) just says the same as

(iv) Γ are tautologically inconsistent.

(5*) *Show that* $\Gamma, \alpha \vDash \bot$ *if and only if* $\Gamma \vDash \neg\alpha$.

This too just falls out of the definitions. By definition of the turnstile,

(i) $\Gamma, \alpha \vDash \bot$

if and only if

(ii) Any valuation of the relevant atoms which makes Γ and α all true together makes $\bot$ true.

But *no* valuation ever makes $\bot$ true. So (ii) can only be true true if and only if

(iii) No valuation of the relevant atoms makes Γ and α all true together.

But (iii) is equivalent to

(iv) Any valuation of the relevant atoms [the ones in Γ and α!] which makes Γ all true together doesn't also make α true.

But valuations settle the value of every wff, so those which don't make α true make it false, and hence make $\neg\alpha$ true. So (iv) is equivalent to

(v) Any valuation of the relevant atoms which makes Γ all true together makes $\neg\alpha$ true

which by the definition of the turnstile is the same claim as

(vi) $\Gamma \vDash \neg\alpha$.