Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 32: QL proofs

In §32.5(d), it was claimed that the following are useful derived rules that could be added to our basic QL proof system to speed up some proofs without changing what can be proved overall:

($\neg\forall$) *Given one of* $\neg\forall\xi\alpha(\xi)$ *and* $\exists\xi\neg\alpha(\xi)$, *you can derive the other.*
($\neg\exists$) *Given one of* $\neg\exists\xi\alpha(\xi)$ *and* $\forall\xi\neg\alpha(\xi)$, *you can derive the other.*

Confirm that these are correct derived rules.

Note first the following two schemas or templates for (parts of) proofs – we just pick up proof ideas from §32.5(b), and move a few negation signs around!

$$\begin{array}{|l l}
\neg\forall\xi\alpha(\xi) & \\
\quad | \; \neg\exists\xi\neg\alpha(\xi) & \text{(Supp)} \\
\quad | \quad | \; \neg\alpha(\delta) & \text{(Supp)} \\
\quad | \quad | \; \exists\xi\neg\alpha(\xi) & (\exists\text{I}) \\
\quad | \quad | \; \bot & \text{(Abs)} \\
\quad | \; \neg\neg\alpha(\delta) & \text{(RAA)} \\
\quad | \; \alpha(\delta) & \text{(DN)} \\
\quad | \; \forall\xi\alpha(\xi) & (\forall\text{I}) \\
\quad | \; \bot & \text{(Abs)} \\
\neg\neg\exists\xi\neg\alpha(\xi) & \text{(RAA)} \\
\exists\xi\neg\alpha(\xi) & \text{(DN)}
\end{array}
\qquad\qquad
\begin{array}{|l l}
\exists\xi\neg\alpha(\xi) & \\
\quad | \; \forall\xi\alpha(\xi) & \text{(Supp)} \\
\quad | \quad | \; \neg\alpha(\delta) & \text{(Supp)} \\
\quad | \quad | \; \alpha(\delta) & (\forall\text{E}) \\
\quad | \quad | \; \bot & \text{(Abs)} \\
\quad | \; \bot & (\forall\text{E}) \\
\neg\forall\xi\alpha(\xi) & \text{(RAA)}
\end{array}$$

So adding the rule that takes us in one step from $\neg\forall\xi$ to $\exists\xi\neg$ (to use an obvious shorthand) doesn't allow us to derive anything that we can't derive the long way, using moves as in the proof outlined on the left. Similarly the companion rule that takes us in one step from $\exists\xi\neg$ to $\neg\forall\xi$ doesn't allow us to derive anything that we can't derive the long way, using moves as in the proof outlined on the right.

Likewise, consider this pair of schemas:

$$\begin{array}{|l l}
\neg\exists\xi\alpha(\xi) & \\
\quad | \; \alpha(\delta) & \text{(Supp)} \\
\quad | \; \exists\xi\alpha(\xi) & (\exists\text{I}) \\
\quad | \; \bot & \text{(Abs)} \\
\neg\alpha(\delta) & \text{(RAA)} \\
\forall\xi\neg\alpha(\xi) & (\forall\text{I})
\end{array}
\qquad\qquad
\begin{array}{|l l}
\forall\xi\neg\alpha(\xi) & \\
\quad | \; \exists\xi\alpha(\xi) & \text{(Supp)} \\
\quad | \quad | \; \alpha(\delta) & \text{(Supp)} \\
\quad | \quad | \; \neg\alpha(\delta) & (\forall\text{E}) \\
\quad | \quad | \; \bot & \text{(Abs)} \\
\quad | \; \bot & (\forall\text{E}) \\
\neg\exists\xi\alpha(\xi) & \text{(RAA)}
\end{array}$$

So again we see that adding the rule that takes us in one step between $\neg\exists\xi$ to $\forall\xi\neg$ (in either direction) doesn't allow us to derive anything that we can't derive using moves already available in our proof system, as in the proofs outlined.

Hence, in summary, both the suggested rules are indeed allowable derived rules in our QL proof system.

And now, then, for the end-of-chapter Exercises ...

(a) *We said that if $\forall\xi\alpha(\xi)$ is a quantified wff, τ is a term, and $\alpha(\tau)$ is the result of replacing every occurrence of ξ in $\alpha(\xi)$ by τ, then $\alpha(\tau)$ is a wff. Why is that true?*

So we don't get our *tau*s in a twist, let's very slightly rephrase the question, using τ' instead!

> We said that if $\forall\xi\alpha(\xi)$ is a quantified wff, τ' is a term, and $\alpha(\tau')$ is the result of replacing every occurrence of ξ in $\alpha(\xi)$ by τ', then $\alpha(\tau')$ is a wff. Why is that true?

Well, let's ask: how can $\forall\xi\alpha(\xi)$ be a wff? §28.6 on the syntax of quantifiers gave us the following one and only rule for forming a universally quantified wff:

> If $\alpha(\tau)$ is a QL wff involving one or more occurrences of the term τ, and $\alpha(\xi)$ is the expression which results from replacing the term τ throughout $\alpha(\tau)$ by a variable ξ new to that wff, then $\forall\xi\alpha(\xi)$ is a QL wff.

So, given wff $\forall\xi\alpha(\xi)$ is a wff, there must at least one wff $\alpha(\tau)$ from which the quantified wff can be formed by our sole rule for building universally quantified wffs.

Take such a $\alpha(\tau)$. What happens if you systematically replace τ in $\alpha(\tau)$ by some *other* term τ'? The expression $\alpha(\tau')$ will be built up from τ' (a proper name or dummy name) and other ingredients *exactly* as the wff $\alpha(\tau)$ is built up from τ (another proper name or dummy name) and those same ingredients. So this will make $\alpha(\tau')$ a wff too. For the status of an expression as a wff obviously doesn't depend on *which* particular proper name or dummy name occurs in a particular place (if one name will do, any other will do too).

Given $\forall\xi\alpha(\xi)$ can be thought of as built by universally quantifying on τ in the instance $\alpha(\tau)$ – i.e. replacing τ with ξ and prefixing the quantifier $\forall\xi$ – we can therefore equally well regard the wff $\forall\xi\alpha(\xi)$ as constructed from the wff (yes, wff!) $\alpha(\tau')$ by universally quantifying on τ' – i.e. replacing τ' with ξ and prefixing the quantifier $\forall\xi$!

So, putting that in reverse, we can regard the wff (yes, wff!) $\alpha(\tau')$ as an instance, the result of result of replacing every occurrence of ξ in $\alpha(\xi)$ by the term τ'. As we said!

(b) *Revisit Exercises 31, and render the informal arguments given there into suitable QL languages, and then provide formal derivations of the conclusions from the premisses. (Don't skip the propositional reasoning in these easy cases!)*

(1) *If Jo can do the exercises, then everyone in the class can do the exercises. Mo is in the class, and can't do the exercises. So Jo can't do the exercises.*

Translation: $(\mathsf{P} \to \forall\mathsf{x}(\mathsf{Cx} \to \mathsf{Dx}))$, $(\mathsf{Cm} \land \neg\mathsf{Dm})$; $\therefore$ $\neg\mathsf{P}$.

For this example and the following ones, revisit the answers to Exercises 31 for informal motivation! Evidently our proof in this case is going to proceed by reductio, assuming P.

(1)	$(\mathsf{P} \to \forall\mathsf{x}(\mathsf{Cx} \to \mathsf{Dx}))$	(Prem)
(2)	$(\mathsf{Cm} \land \neg\mathsf{Dm})$	(Prem)
(3)	$\quad \mathsf{P}$	(Supp)
(4)	$\quad \forall\mathsf{x}(\mathsf{Cx} \to \mathsf{Dx})$	(MP 3, 1)
(5)	$\quad (\mathsf{Cm} \to \mathsf{Dm})$	($\forall$E 4)
(6)	$\quad \mathsf{Cm}$	($\land$E 2)
(7)	$\quad \neg\mathsf{Dm}$	($\land$E 2)
(8)	$\quad \mathsf{Dm}$	(MP 6, 5)
(9)	$\quad \bot$	(Abs 8, 7)
(10)	$\neg\mathsf{P}$	(RAA 3–9)

(2) *No whales are fish. So no fish are whales.*

We can translate ‘no’ propositions in two ways, using universal or existential quantification, giving us four alternative formal renditions of the argument. Let’s look at two:

$$\forall x(Wx \to \neg Fx) \ \therefore\ \forall x(Fx \to \neg Wx).$$
$$\neg\exists x(Wx \land Fx) \ \therefore\ \neg\exists x(Fx \land Wx).$$

For the first version we can argue in outline as follows:

(1)	$\mid\ \forall x(Wx \to \neg Fx)$	(Prem)
(2)	$\mid\ (Wa \to \neg Fa)$	($\land$E, 1)
(3)	$\mid\ (Fa \to \neg Wa)$	(PL 2)
(4)	$\mid\ \forall x(Fx \to \neg Wx)$	($\forall$I 3)

where at the third step we compress some reasoning with the connectives. The basic plan, then, is that we take an arbitrary representative object, and apply (1); we fiddle around to get another claim about the same arbitrary representative; and generalize to get (4). In detail:

(1)	$\mid\ \forall x(Wx \to \neg Fx)$	(Prem)
(2)	$\mid\ (Wa \to \neg Fa)$	($\land$E, 1)
(3)	$\mid\ \mid\ Fa$	(Supp)
(4)	$\mid\ \mid\ \mid\ Wa$	(Supp)
(5)	$\mid\ \mid\ \mid\ \neg Fa$	(MP 4, 2)
(6)	$\mid\ \mid\ \mid\ \bot$	(Abs 3, 5)
(7)	$\mid\ \mid\ \neg Wa$	(RAA 4–6)
(8)	$\mid\ (Fa \to \neg Wa)$	(CP 3–7)
(9)	$\mid\ \forall x(Fx \to \neg Wx)$	($\forall$I 8)

For the second version, we obviously want a reductio argument of this shape:

$\mid\ \neg\exists x(Wx \land Fx)$	(Prem)
$\mid\ \mid\ \exists x(Fx \land Wx)$	(Supp)
$\mid\ \mid\ \vdots$	
$\mid\ \mid\ \bot$	
$\mid\ \neg\exists x(Fx \land Wx)$	(RAA)

So how are we going to fill in the dots? We will need to use the existentially quantified supposition: so evidently we will need to suppose an instance of it with an eye to invoking the rule ($\exists$E). But with that hint, the proof more or less writes itself:

(1)	$\mid\ \neg\exists x(Wx \land Fx)$	(Prem)
(2)	$\mid\ \mid\ \exists x(Fx \land Wx)$	(Supp)
(3)	$\mid\ \mid\ \mid\ (Fa \land Wa)$	(Supp)
(4)	$\mid\ \mid\ \mid\ Wa$	($\land$E 3)
(5)	$\mid\ \mid\ \mid\ Fa$	($\land$E 3)
(6)	$\mid\ \mid\ \mid\ (Wa \land Fa)$	($\land$I 4, 5)
(7)	$\mid\ \mid\ \mid\ \exists x(Wx \land Fx)$	($\lor$I 6)
(8)	$\mid\ \mid\ \mid\ \bot$	(Abs 7, 1)
(9)	$\mid\ \mid\ \bot$	($\exists$E 2, 3–8)
(10)	$\mid\ \neg\exists x(Fx \land Wx)$	(RAA 2–9)

(3) *All leptons have half-integer spin. All electrons are leptons. So all electrons have half-integer spin.*

Translation: $\forall x(Lx \to Hx),\ \forall x(Ex \to Lx)\ \therefore\ \forall x(Ex \to Hx)$.

We know we just need to think about an arbitrary item in the domain; to cheerfully muddle up our languages, the premisses tell us that if it is E it is L and hence H. So we can argue …

(1)	$\mid\ \forall x(Lx \to Hx)$	(Prem)
(2)	$\mid\ \underline{\forall x(Ex \to Lx)}$	(Prem)
(3)	$\mid\ (La \to Ha)$	(∀E 1)
(4)	$\mid\ (Ea \to La)$	(∀E 2)
(5)	$\mid\ \quad \mid\ \underline{Ea}$	(Supp)
(6)	$\mid\ \quad \mid\ La$	(MP 5, 4)
(7)	$\mid\ \quad \mid\ Ha$	(MP 6, 3)
(8)	$\mid\ (Ea \to Ha)$	(CP 3–7)
(9)	$\mid\ \forall x(Ex \to Hx)$	(∀I 8 – permissible since a isn't in any still-live assumption)

(4) *Some chaotic attractors are not fractals. Every Cantor set is a fractal. Hence some chaotic attractors are not Cantor sets.*

Translation: $\exists x(Ax \wedge \neg Fx)$, $\forall x(Cx \to Fx)$ $\therefore$ $\exists x(Ax \wedge \neg Cx)$.

The shape of the proof we need is plain:

$\mid\ \exists x(Ax \wedge \neg Fx)$	(Prem)
$\mid\ \underline{\forall x(Cx \to Fx)}$	(Prem)
$\mid\ \quad \mid\ \underline{(Aa \wedge \neg Fa)}$	(Supp)
$\mid\ \quad \mid\ \vdots$	
$\mid\ \quad \mid\ \exists x(Ax \wedge \neg Cx)$	
$\mid\ \exists x(Ax \wedge \neg Cx)$	(∃E)

That's because we are evidently going to need to use the first, existentially quantified, premiss – and that will involve making a supposition as at the third line. And how can we derive the penultimate line? Presumably from an instance $(Aa \wedge \neg Ca)$:

$\mid\ \exists x(Ax \wedge \neg Fx)$	(Prem)
$\mid\ \underline{\forall x(Cx \to Fx)}$	(Prem)
$\mid\ \quad \mid\ \underline{(Aa \wedge \neg Fa)}$	(Supp)
$\mid\ \quad \mid\ (Ca \to Fa)$	(∀E 2)
$\mid\ \quad \mid\ \vdots$	
$\mid\ \quad \mid\ (Aa \wedge \neg Ca)$	(by PL reasoning)
$\mid\ \quad \mid\ \exists x(Ax \wedge \neg Cx)$	(∨I)
$\mid\ \exists x(Ax \wedge \neg Cx)$	(∃E)

Filling in the dots, though, is easy propositional reasoning!

(1)	$\mid\ \exists x(Ax \wedge \neg Fx)$	(Prem)
(2)	$\mid\ \underline{\forall x(Cx \to Fx)}$	(Prem)
(3)	$\mid\ \quad \mid\ \underline{(Aa \wedge \neg Fa)}$	(Supp)
(4)	$\mid\ \quad \mid\ (Ca \to Fa)$	(∀E 2)
(5)	$\mid\ \quad \mid\ Aa$	(∧E 3)
(6)	$\mid\ \quad \mid\ \neg Fa$	(∧E 3)
(7)	$\mid\ \quad \mid\ \quad \mid\ \underline{Ca}$	(Supp)
(8)	$\mid\ \quad \mid\ \quad \mid\ Fa$	(MP 7, 4)
(9)	$\mid\ \quad \mid\ \quad \mid\ \bot$	(Abs 8, 6)
(10)	$\mid\ \quad \mid\ \neg Ca$	(RAA 7–9)
(11)	$\mid\ \quad \mid\ (Aa \wedge \neg Ca)$	(∧I 5, 10)
(12)	$\mid\ \quad \mid\ \exists x(Ax \wedge \neg Cx)$	(∨I 11)
(13)	$\mid\ \exists x(Ax \wedge \neg Cx)$	(∃E 1, 3–12)

(5) *Some philosophers are logicians. All logicians are rational people. No rational person is a flat-earther. Therefore some philosophers are not flat-earthers.*

Translation: $\exists x(Px \wedge Lx)$, $\forall x(Lx \to Rx)$, $\forall x(Rx \to \neg Fx)$ $\therefore$ $\exists x(Px \wedge \neg Fx)$.

To make use of the first premiss, we'll want a proof of the shape

$\mid\ \exists x(Px \wedge Lx)$	(Prem)
$\mid\ \forall x(Lx \to Rx)$	(Prem)
$\mid\ \underline{\forall x(Rx \to \neg Fx)}$	(Prem)
$\mid\ \mid\ \underline{(Pa \wedge La)}$	(Supp)
$\mid\ \mid\ \vdots$	
$\mid\ \mid\ \exists x(Px \wedge \neg Fx)$	
$\mid\ \exists x(Px \wedge \neg Fx)$	($\exists$E from first premiss and the subproof)

So filling in the details in the obvious way, we get

(1)	$\mid\ \exists x(Px \wedge Lx)$	(Prem)
(2)	$\mid\ \forall x(Lx \to Rx)$	(Prem)
(3)	$\mid\ \underline{\forall x(Rx \to \neg Fx)}$	(Prem)
(4)	$\mid\ \mid\ \underline{(Pa \wedge La)}$	(Supp)
(5)	$\mid\ \mid\ (La \to Ra)$	($\forall$E 2)
(6)	$\mid\ \mid\ (Ra \to \neg Fa)$	($\forall$E 3)
(7)	$\mid\ \mid\ Pa$	($\wedge$E 4)
(8)	$\mid\ \mid\ La$	($\wedge$E 4)
(9)	$\mid\ \mid\ Ra$	(MP 8, 5)
(10)	$\mid\ \mid\ \neg Fa$	(MP 9, 6)
(11)	$\mid\ \mid\ (Pa \wedge \neg Fa)$	($\wedge$I 7, 10)
(12)	$\mid\ \mid\ \exists x(Px \wedge \neg Fx)$	($\exists$I 11)
(13)	$\mid\ \exists x(Px \wedge \neg Fx)$	($\exists$E 1, 4–12)

What if we'd used the other rendition of the 'no' premiss, so we translate the argument

Translation: $\exists x(Px \wedge Lx)$, $\forall x(Lx \to Rx)$, $\neg\exists x(Rx \wedge Fx)$ $\therefore$ $\exists x(Px \wedge \neg Fx)$?

Then here's a derivation with the same basic shape, but with the details filled differently.

(1)	$\mid\ \exists x(Px \wedge Lx)$	(Prem)
(2)	$\mid\ \forall x(Lx \to Rx)$	(Prem)
(3)	$\mid\ \underline{\neg\exists x(Rx \wedge Fx)}$	(Prem)
(4)	$\mid\ \mid\ \underline{(Pa \wedge La)}$	(Supp)
(5)	$\mid\ \mid\ (La \to Ra)$	($\forall$E 2)
(6)	$\mid\ \mid\ Pa$	($\wedge$E 4)
(7)	$\mid\ \mid\ La$	($\wedge$E 4)
(8)	$\mid\ \mid\ Ra$	(MP 7, 5)
(9)	$\mid\ \mid\ \mid\ \underline{Fa}$	(Supp)
(10)	$\mid\ \mid\ \mid\ (Ra \wedge Fa)$	($\wedge$I 8, 9)
(11)	$\mid\ \mid\ \mid\ \exists x(Rx \wedge Fx)$	($\exists$I 10)
(12)	$\mid\ \mid\ \mid\ \bot$	(Abs 11, 2)
(13)	$\mid\ \mid\ \neg Fa$	(RAA 9–12)
(14)	$\mid\ \mid\ (Pa \wedge \neg Fa)$	($\wedge$I 7, 10)
(15)	$\mid\ \mid\ \exists x(Px \wedge \neg Fx)$	($\exists$I 11)
(16)	$\mid\ \exists x(Px \wedge \neg Fx)$	($\exists$E 1, 4–12)

A general point arising. If you want to make use of a premiss, assumption, or interim conclusion which is a *negated* existential proposition of the form $\neg\exists\xi\alpha(\xi)$, very often the thing to do is aim to prove the unnegated $\exists\xi\alpha(\xi)$ from some assumption β. And then you'll have – as here – a reductio proof of $\neg\beta$. (Or "cheat" and use the $(\neg\exists)$ derived rule!!)

(6) *All lions and tigers are dangerous animals. Dangerous animals should be avoided. Leo is a lion. So Leo should be avoided.*

We indicated three possible regimentations of the first premiss in the answers to Exercises 31; let's formalize two of them to give the following two arguments:

$\forall x((Lx \to Dx) \land (Tx \to Dx)),\ \forall x(Dx \to Ax)\ Lm\ \therefore\ Am$

$\forall x((Lx \lor Tx) \to Dx),\ \forall x(Dx \to Ax)\ Lm\ \therefore\ Am$

The proofs are easy, either way:

(1)	$\forall x((Lx \to Dx) \land (Tx \to Dx))$	(Prem)
(2)	$\forall x(Dx \to Ax)$	(Prem)
(3)	Lm	(Prem)
(4)	$((Lm \to Dm) \land (Tm \to Dm))$	($\forall$E 1)
(5)	$(Dm \to Am)$	($\forall$E 2)
(6)	$(Lm \to Dm)$	($\land$E 4)
(7)	Dm	(MP 3, 6)
(8)	Am	(MP 7, 5)

(1)	$\forall x((Lx \lor Tx) \to Dx)$	(Prem)
(2)	$\forall x(Dx \to Ax)$	(Prem)
(3)	Lm	(Prem)
(4)	$((Lm \lor Tm) \to Dm)$	($\forall$E 1)
(5)	$(Dm \to Am)$	($\forall$E 2)
(6)	$(Lm \lor Tm)$	($\lor$I 3)
(7)	Dm	(MP 6, 4)
(8)	Am	(MP 7, 5)

(c) *Also translate the following into suitable* QL *languages, and again provide formal derivations of the conclusions from the premisses:*

(1) *If Jones is a bad philosopher, then some Welsh speaker is irrational; but every Welsh speaker is rational; hence Jones is not a bad philosopher.*

Translation: $(P \to \exists x(Wx \land \neg Rx)),\ \forall x(Wx \to Rx)\ \therefore\ \neg P$

We are not interested here in the internal structure of the proposition 'Jones is a bad philosopher' so we can use a single letter (a zero-place predicate!) to represent it.

Since we want to prove $\neg P$, the obvious strategy for a formal proof is to assume 'P' and aim for a contradiction. Let's do it! So the proof will have this shape:

$(P \to \exists x(Wx \land \neg Rx))$	(Prem)
$\forall x(Wx \to Rx)$	(Prem)
$\quad P$	(Supp)
$\quad \exists x(Wx \land \neg Rx)$	(MP)
$\quad \vdots$	
$\quad \bot$	
$\neg P$	(RAA)

How can we use the existential quantification at the fourth line? We obviously need to make a new supposition, supposing an instance [using a dummy name] is true, with a view to an argument by (∨E). Which suggests the argument, filled out a little more, will look like this:

$$
\begin{array}{ll}
\mid (\mathsf{P} \to \exists \mathsf{x}(\mathsf{Wx} \land \neg \mathsf{Rx})) & \text{(Prem)} \\
\mid \underline{\forall \mathsf{x}(\mathsf{Wx} \to \mathsf{Rx})} & \text{(Prem)} \\
\mid \quad \mid \underline{\mathsf{P}} & \text{(Supp)} \\
\mid \quad \mid \exists \mathsf{x}(\mathsf{Wx} \land \neg \mathsf{Rx}) & \text{(MP)} \\
\mid \quad \mid \quad \mid \underline{(\mathsf{Wa} \land \mathsf{Ra})} & \text{(Supp)} \\
\mid \quad \mid \quad \mid \vdots & \\
\mid \quad \mid \quad \mid \bot & \\
\mid \quad \mid \bot & \text{(}\lor\text{E from } \exists \mathsf{x}(\mathsf{Wx} \land \neg \mathsf{Rx}) \text{ and subproof)} \\
\mid \neg \mathsf{P} & \text{(RAA)}
\end{array}
$$

But now the proof is quickly finished. We obviously need to use the second premiss, so instantiate that using the term 'a', and our desired contradiction is almost immediate:

$$
\begin{array}{rll}
1 & \mid (\mathsf{P} \to \exists \mathsf{x}(\mathsf{Wx} \land \neg \mathsf{Rx})) & \text{(Prem)} \\
2 & \mid \underline{\forall \mathsf{x}(\mathsf{Wx} \to \mathsf{Rx})} & \text{(Prem)} \\
3 & \mid \quad \mid \underline{\mathsf{P}} & \text{(Supp)} \\
4 & \mid \quad \mid \exists \mathsf{x}(\mathsf{Wx} \land \neg \mathsf{Rx}) & \text{(MP 3, 1)} \\
5 & \mid \quad \mid \quad \mid \underline{(\mathsf{Wa} \land \neg \mathsf{Ra})} & \text{(Supp)} \\
6 & \mid \quad \mid \quad \mid \mathsf{Wa} & \text{(}\land\text{E 5)} \\
7 & \mid \quad \mid \quad \mid \neg \mathsf{Ra} & \text{(}\land\text{E 5)} \\
8 & \mid \quad \mid \quad \mid (\mathsf{Wa} \to \mathsf{Ra}) & \text{(}\forall\text{E 2)} \\
9 & \mid \quad \mid \quad \mid \mathsf{Ra} & \text{(MP 6, 8)} \\
10 & \mid \quad \mid \quad \mid \bot & \text{(Abs 9, 7)} \\
11 & \mid \quad \mid \bot & \text{(}\exists\text{E 4, 5–10)} \\
12 & \mid \neg \mathsf{P} & \text{(RAA)}
\end{array}
$$

(2) *Everyone is such that, if they admire Ludwig, then the world has gone mad. Therefore, if someone admires Ludwig, the world has gone mad.*

Translation: $\forall \mathsf{x}(\mathsf{Lx} \to \mathsf{P}) \ \therefore \ (\exists \mathsf{x}\, \mathsf{Lx} \to \mathsf{P})$

Here, the quantifier runs over the relevant people. We need expose no internal structure in the predicate 'admires Ludwig', so a simple unary predicate will suffice. And again we are not interested here in the internal structure of the proposition 'the world has gone mad' so we can use a single letter to represent it.

Here's overall proof shape we will evidently need (what else can we do but assume the conclusion is false and aim for a reductio?)

$$
\begin{array}{ll}
\mid \underline{\forall \mathsf{x}(\mathsf{Lx} \to \mathsf{P})} & \text{(Prem)} \\
\mid \quad \mid \underline{\neg(\exists \mathsf{x}\, \mathsf{Lx} \to \mathsf{P})} & \text{(Supp)} \\
\mid \quad \mid \quad \vdots & \\
\mid \quad \mid \bot & \\
\mid \neg\neg(\exists \mathsf{x}\, \mathsf{Lx} \to \mathsf{P}) & \text{(RAA)} \\
\mid (\exists \mathsf{x}\, \mathsf{Lx} \to \mathsf{P}) & \text{(DN)}
\end{array}
$$

Now, let's allow ourselves to skip the detail of PL reasoning; we know that from a wff of the from $\neg(\alpha \to \beta)$ we can derive both α and $\neg\beta$. So apply that to our supposition, and we get:

$$
\begin{array}{llll}
| & \forall x(Lx \to P) & & \text{(Prem)} \\
| & \quad | & \neg(\exists x\, Lx \to P) & \text{(Supp)} \\
| & \quad | & \exists x\, Lx & \text{(by PL reasoning from the supposition)} \\
| & \quad | & \neg P & \text{(by PL reasoning from the supposition)} \\
| & \quad | & \quad \vdots & \\
| & \quad | & \bot & \\
| & \neg\neg(\exists x\, Lx \to P) & & \text{(RAA)} \\
| & (\exists x\, Lx \to P) & & \text{(DN)}
\end{array}
$$

So now how can we use the existential quantification on the third line? We will need to set an inference by (∃E) by taking an instance of it as a further supposition. So we will now expect to get

$$
\begin{array}{lllll}
| & \forall x(Lx \to P) & & & \text{(Prem)} \\
| & \quad | & \neg(\exists x\, Lx \to P) & & \text{(Supp)} \\
| & \quad | & \exists x\, Lx & & \text{(by PL reasoning from the supposition)} \\
| & \quad | & \neg P & & \text{(by PL reasoning from the supposition)} \\
| & \quad | & \quad | & La & \text{(Supp)} \\
| & \quad | & \quad | & \vdots & \\
| & \quad | & \quad | & \bot & \\
| & \quad | & \bot & & \text{(}\exists\text{E using wff at the third line and the subproof)} \\
| & \neg\neg(\exists x\, Lx \to P) & & & \text{(RAA)} \\
| & (\exists x\, Lx \to P) & & & \text{(DN)}
\end{array}
$$

And now linking the new supposition to the initial premiss, the proof finishes itself very easily!

$$
\begin{array}{llllll}
(1) & | & \forall x(Lx \to P) & & & \text{(Prem)} \\
(2) & | & \quad | & \neg(\exists x\, Lx \to P) & & \text{(Supp)} \\
(3) & | & \quad | & \exists x\, Lx & & \text{(PL 2)} \\
(4) & | & \quad | & \neg P & & \text{(PL 2)} \\
(5) & | & \quad | & \quad | & La & \text{(Supp)} \\
(6) & | & \quad | & \quad | & (La \to P) & (\forall\text{E 1}) \\
(7) & | & \quad | & \quad | & P & \text{(MP 5, 6)} \\
(8) & | & \quad | & \quad | & \bot & \text{(Abs 7, 4)} \\
(9) & | & \quad | & \bot & & (\exists\text{E 3, 4–8}) \\
(10) & | & \neg\neg(\exists x\, Lx \to P) & & & \text{(RAA 2–9)} \\
(11) & | & (\exists x\, Lx \to P) & & & \text{(DN 10)}
\end{array}
$$

(3) *Jack is taller than Jill. Someone is taller than Jack. If a first person is taller than a second, and the second is taller than a third, then the first person is taller than the third. Hence someone is taller than both Jack and Jill.*

Translation: $Tmn,\ \exists xTxm,\ \forall x\forall y\forall z((Txy \land Tyz) \to Txz)\ \therefore\ \exists x(Lxm \land Txn)$

Arguing informally, the second premiss tells us that there is someone (Alice, if you like) who is taller than Jack. So Alice is taller than Jack who is taller than Jill. By the third premiss, it follows that Alice is taller than Jill. So indeed there is someone (Alice!) who is is taller than both Jack and Jill.

Now we turn that informal line of argument into a formal deduction. Do note, by the way, that formally our (∀I) only allows us to *instantiate one quantifier at a time* from a wff like the second premiss.

(1)	Tmn	(Prem)
(2)	$\exists\mathsf{xTxm}$	(Prem)
(3)	$\forall\mathsf{x}\forall\mathsf{y}\forall\mathsf{z}((\mathsf{Txy} \land \mathsf{Tyz}) \to \mathsf{Txz})$	(Prem)
(4)	$\quad \mathsf{Tam}$	(Supp)
(5)	$\quad (\mathsf{Tam} \land \mathsf{Tmn})$	($\land$I 4, 3)
(6)	$\quad \forall\mathsf{y}\forall\mathsf{z}((\mathsf{Tay} \land \mathsf{Tyz}) \to \mathsf{Taz})$	($\forall$E 3)
(7)	$\quad \forall\mathsf{z}((\mathsf{Tam} \land \mathsf{Tmz}) \to \mathsf{Taz})$	($\forall$E 5)
(8)	$\quad ((\mathsf{Tam} \land \mathsf{Tmn}) \to \mathsf{Tan})$	($\forall$E 5)
(9)	$\quad \mathsf{Tan}$	(MP 5, 8)
(10)	$\quad (\mathsf{Tam} \land \mathsf{Tan})$	($\land$E 10)
(11)	$\quad \exists\mathsf{x}(\mathsf{Lxm} \land \mathsf{Txn})$	($\exists$I 9)
(12)	$\exists\mathsf{x}(\mathsf{Lxm} \land \mathsf{Txn})$	($\exists$E 3, 4–10)

(4) *Every logician admires Gödel. Whoever admires someone is not without feeling. Hence no logician is without feeling.*

Using an obvious glossary, we have the following possible translations:

$\forall\mathsf{x}(\mathsf{Lx} \to \mathsf{Axg}),\ \forall\mathsf{x}(\exists\mathsf{yAxy} \to \neg\mathsf{Wx})\ \therefore\ \forall\mathsf{x}(\mathsf{Lx} \to \neg\mathsf{Wx})$

$\forall\mathsf{x}(\mathsf{Lx} \to \mathsf{Axg}),\ \forall\mathsf{x}(\exists\mathsf{yAxy} \to \neg\mathsf{Wx})\ \therefore\ \neg\exists\mathsf{x}(\mathsf{Lx} \land \mathsf{Wx})$

In the first case, we need to prove a universal generalization. Aim to establish an arbitrary instance of it, $(\mathsf{La} \to \neg\mathsf{Wa})$. That's a conditional, so suppose La and aim for $\neg\mathsf{Wa}$ …

(1)	$\forall\mathsf{x}(\mathsf{Lx} \to \mathsf{Axg})$	(Prem)
(2)	$\forall\mathsf{x}(\exists\mathsf{yAxy} \to \neg\mathsf{Wx})$	(Prem)
(3)	$\quad \mathsf{La}$	(Supp)
(4)	$\quad (\mathsf{La} \to \mathsf{Aag})$	($\forall$E 1)
(5)	$\quad \mathsf{Aag}$	(MP 3, 4)
(6)	$\quad (\exists\mathsf{yAay} \to \neg\mathsf{Wa})$	($\forall$E 2)
(7)	$\quad \exists\mathsf{yAay}$	($\exists$I 5)
(8)	$\quad \neg\mathsf{Wa}$	(MP 7, 6)
(9)	$(\mathsf{La} \to \neg\mathsf{Wa})$	(CP 3–8)
(10)	$\forall\mathsf{x}(\mathsf{Lx} \to \neg\mathsf{Wx})$	($\forall$I 9)

For the second version, we argue

(1)	$\forall\mathsf{x}(\mathsf{Lx} \to \mathsf{Axg})$	(Prem)
(2)	$\forall\mathsf{x}(\exists\mathsf{yAxy} \to \neg\mathsf{Wx})$	(Prem)
(3)	$\quad \exists\mathsf{x}(\mathsf{Lx} \land \mathsf{Wx})$	(Supp)
(4)	$\quad\quad (\mathsf{La} \land \mathsf{Wa})$	(Supp)
(5)	$\quad\quad \mathsf{La}$	($\land$E 4)
(6)	$\quad\quad \mathsf{Wa}$	($\land$E 4)
(7)	$\quad\quad (\mathsf{La} \to \mathsf{Aag})$	($\forall$E 1)
(8)	$\quad\quad \mathsf{Aag}$	(MP 5, 7)
(9)	$\quad\quad \exists\mathsf{yAay}$	($\exists$I 8)
(10)	$\quad\quad (\exists\mathsf{yAay} \to \neg\mathsf{Wa})$	($\forall$E 2)
(11)	$\quad\quad \neg\mathsf{Wa}$	(MP 7, 6)
(12)	$\quad\quad \bot$	(Abs 6, 11)
(13)	$\quad \bot$	($\exists$E 3, 4–12)
(14)	$\neg\exists\mathsf{x}(\mathsf{Lx} \land \mathsf{Wx})$	(RAA 3–13)

(5) *Either not everyone liked the cake or someone baked an excellent cake. If I'm right, then whoever bakes an excellent cake ought to be proud. So if everyone liked the cake and I'm right, then someone ought to be proud.*

Note that the internal structure of 'everyone liked the cake' does no work at all, so we might as well represent it as an unstructured proposition E. Again the internal structure of the predicate 'baked an excellent cake' does no work: so we might as well represent it with an unstructured unary predicate B. Which gives us the rendition

Translation: $(\neg\mathsf{E} \lor \exists\mathsf{y}\mathsf{B}\mathsf{y}),\ (\mathsf{R} \to \forall\mathsf{x}(\mathsf{B}\mathsf{x} \to \mathsf{P}\mathsf{x}))\ \therefore\ ((\mathsf{E} \land \mathsf{R}) \to \exists\mathsf{x}\mathsf{P}\mathsf{x})$

To prove the conditional conclusion, assume the antecedent. Using disjunctive syllogism gives us $\exists\mathsf{y}\mathsf{B}\mathsf{y}$ from the first premiss. Using modus ponens gives us $\forall\mathsf{x}(\mathsf{B}\mathsf{x} \to \mathsf{P}\mathsf{x})$ from the second premiss. But those two quantified propositions yield $\exists\mathsf{x}\mathsf{P}\mathsf{x}$ in a now familiar way. So in gory detail . . .

(1)	$(\neg\mathsf{E} \lor \exists\mathsf{y}\mathsf{B}\mathsf{y})$	(Prem)
(2)	$(\mathsf{R} \to \forall\mathsf{x}(\mathsf{B}\mathsf{x} \to \mathsf{P}\mathsf{x}))$	(Prem)
(3)	$\quad(\mathsf{E} \land \mathsf{R})$	(Supp)
(4)	$\quad\mathsf{E}$	(∧E 3)
(5)	$\quad\mathsf{R}$	(∧E 3)
(6)	$\quad\quad\neg\mathsf{E}$	(Supp)
(7)	$\quad\quad\bot$	(Abs 4, 6)
(8)	$\quad\quad\exists\mathsf{y}\mathsf{B}\mathsf{y}$	(EFQ 7)
(9)	$\quad\quad\exists\mathsf{y}\mathsf{B}\mathsf{y}$	(Supp)
(10)	$\quad\quad\exists\mathsf{y}\mathsf{B}\mathsf{y}$	(Iter 9)
(11)	$\quad\exists\mathsf{y}\mathsf{B}\mathsf{y}$	(∨E 1, 6–8, 9–10)
(12)	$\quad\forall\mathsf{x}(\mathsf{B}\mathsf{x} \to \mathsf{P}\mathsf{x})$	(MP 4, 2)
(13)	$\quad\quad\mathsf{Ba}$	(Supp)
(14)	$\quad\quad(\mathsf{Ba} \to \mathsf{Pa})$	(∀E 12)
(15)	$\quad\quad\mathsf{Pa}$	(MP 13, 14)
(16)	$\quad\quad\exists\mathsf{x}\mathsf{P}\mathsf{x}$	(∃I 15)
(17)	$\quad\exists\mathsf{x}\mathsf{P}\mathsf{x}$	(∃E 11, 13–16)
(18)	$((\mathsf{E} \land \mathsf{R}) \to \exists\mathsf{x}\mathsf{P}\mathsf{x})$	(CP 3–17)

Also translate the following into suitable QL *languages and show that they are logical truths by deriving them as theorems:*

(6) *Everyone is either a logician or not a logician.*

We want to prove something of the form $\forall\mathsf{x}(\mathsf{L}\mathsf{x} \lor \neg\mathsf{L}\mathsf{x})$. We just (i) prove $(\mathsf{La} \land \neg\mathsf{La})$ from no assumptions and then (ii) universally generalize. Step (i) is bookwork. Step (ii) is now trivial.

(1)		
(2)	$\quad\neg(\mathsf{La} \lor \neg\mathsf{La})$	(Supp)
(3)	$\quad\quad\mathsf{La}$	(Supp)
(4)	$\quad\quad(\mathsf{La} \lor \neg\mathsf{La})$	(∨I 3)
(5)	$\quad\quad\bot$	(Abs 4, 2)
(6)	$\quad\neg\mathsf{La}$	(RAA 3–5)
(7)	$\quad(\mathsf{La} \lor \neg\mathsf{La})$	(∨I 6)
(8)	$\quad\bot$	(Abs 7, 2)
(9)	$\neg\neg(\mathsf{La} \lor \neg\mathsf{La})$	(RAA 2–8)
(10)	$(\mathsf{La} \lor \neg\mathsf{La})$	(DN 9)
(11)	$\forall\mathsf{x}(\mathsf{L}\mathsf{x} \lor \neg\mathsf{L}\mathsf{x})$	(∀I 10)

We've said that we are standardly interested in QL proofs which start from zero or more *sentences* and end with a *sentence* (because it is only sentences without dangling dummy names that have a given fixed interpretation in a language). But when we come to officially describing proofs, it is convenient to allow them to go from wffs to wffs more generally – so a derivation that stopped at line 10 would still count as well-formed.

(7) *It's not the case that all logicians are wise while someone is an unwise logician.*

Translation: $(\neg(\forall x(Lx \to Wx) \land \exists x(Lx \land \neg Wx))$

A simple reductio argument does the trick!

(1)	$\mid$	
(2)	$\mid\quad\mid\ (\forall x(Lx \to Wx) \land \exists x(Lx \land \neg Wx))$	(Supp)
(3)	$\mid\quad\mid\ \forall x(Lx \to Wx)$	(∧E 21)
(4)	$\mid\quad\mid\ \exists x(Lx \land \neg Wx)$	(∧E 2)
(5)	$\mid\quad\mid\quad\mid\ (La \land \neg Wa)$	(Supp)
(6)	$\mid\quad\mid\quad\mid\ La$	(∧E 5)
(7)	$\mid\quad\mid\quad\mid\ \neg Wa$	(∧E 5)
(8)	$\mid\quad\mid\quad\mid\ (La \to Wa)$	(∀E 2)
(9)	$\mid\quad\mid\quad\mid\ Wa$	(MP 6, 8)
(10)	$\mid\quad\mid\quad\mid\ \bot$	(Abs 9, 7)
(11)	$\mid\quad\mid\ \bot$	(∨E 4, 5–10)
(12)	$\mid\ \neg(\forall x(Lx \to Wx) \land \exists x(Lx \land \neg Wx))$	(RAA 2–11)

(8) *Everyone has someone whom either they love (despite that person loving themself!) or they don't love (unless that person doesn't love themself!).*

'Despite' we treat as a colourful conjunction. The translational hurdle here is 'unless'. But recall: 'A unless B' tells us that 'if *not-B* then A', or equivalently 'either B or A'. Using the first, and going via the Loglish

> (Everyone x is such that)(there is someone y such that)[either (x loves y and y loves y) or (if it isn't the case that y doesn't love y, then x doesn't love y)]

which gives us

$$\forall x\exists y((Lxy \land Lyy) \lor (\neg\neg Lyy \to \neg Lxy))$$

How can this be a theorem? Presumably if it is, the last two lines of the proof will be (for some dummy name, say 'a')

$\mid\ \exists y((Lay \land Lyy) \lor (\neg\neg Lyy \to \neg Lay))$	
$\mid\ \forall x\exists y((Lxy \land Lyy) \lor (\neg\neg Lyy \to \neg Lxy))$	(∀I)

And how can we prove that existential wff (from no premisses!). Reductio is our only hope. So we want a proof with this shape

$\mid$	
$\mid\quad\mid\ \neg\exists y((Lay \land Lyy) \lor (\neg\neg Lyy \to \neg Lay))$	(Supp)
$\mid\quad\mid\ \vdots$	
$\mid\quad\mid\ \bot$	
$\mid\ \neg\neg\exists y((Lay \land Lyy) \lor (\neg\neg Lyy \to \neg Lay))$	(RAA)
$\mid\ \exists y((Lay \land Lyy) \lor (\neg\neg Lyy \to \neg Lay))$	(DN)
$\mid\ \forall x\exists y((Lxy \land Lyy) \lor (\neg\neg Lyy \to \neg Lxy))$	(∀I)

How do we fill in the dots? As before, we can make use of a negated existential by assuming something that implies the unnegated version of that existential, and apply reductio. Here's one way:

(1)		
(2)	$\quad \neg\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \to \neg Lay))$	(Supp)
(3)	$\quad\quad ((Laa \wedge Laa) \vee (\neg\neg Laa \to \neg Laa))$	(Supp)
(4)	$\quad\quad \exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \to \neg Lay))$	($\exists$I 3)
(5)	$\quad\quad \bot$	(5, 2)
(6)	$\quad \neg((Laa \wedge Laa) \vee (\neg\neg Laa \to \neg Laa))$	(RAA 3–5)
(7)	$\quad \bot$	(PL 6)
(8)	$\neg\neg\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \to \neg Lay))$	(RAA 2–7)
(9)	$\exists y((Lay \wedge Lyy) \vee (\neg\neg Lyy \to \neg Lay))$	(DN 8)
(10)	$\forall x\exists y((Lxy \wedge Lyy) \vee (\neg\neg Lyy \to \neg Lxy))$	($\forall$I 9)

The cunning move here is at line (5) – remembering that when we existentially generalize on a term like a, we do *not* have to generalize on every occurrence of a. We could also have got from (2) to (6) by applying the derived rule that takes us from $\neg\exists y$ to $\forall y\neg$ and then instantiating the universal quantifier with a.

You can be trusted by now to fill in the PL reasoning from (6) to (7)! And if you'd used the other, disjunctive, rendition of 'unless' to translate the argument, the derivation would have gone very similarly.

(d) *Give proofs to warrant the following inferences (compare §29.6):*

(1) $(P \wedge \forall x\, Fx) \;\therefore\; \forall x(P \wedge Fx)$

This is very straightforward! We have to unpack the conjunctive premiss. We then need to aim to prove something like $(P \wedge Fa)$ so we can universally quantify to get the desired conclusion. We've already got P. How can we get Fa?

(1)	$(P \wedge \forall x\, Fx)$	(Prem)
(2)	P	($\wedge$E 1)
(3)	$\forall x\, Fx$	($\wedge$E 1)
(4)	Fa	($\forall$E 3)
(5)	$(P \wedge Fa)$	($\wedge$I 2, 4)
(6)	$\forall x(P \wedge Fx)$	($\forall$I 5)

(2) $(\exists y\, Fy \vee P) \;\therefore\; \exists y(Fy \vee P)$

We'll need to argue by cases. So, first case, we show $\exists y Fy$ implies $\exists y(Fy \vee P)$. This will be a pretty simple ($\exists$E) argument.

Then, second case, we need to show P also implies $\exists y(Fy \vee P)$. But that only takes one intermediate step!

(1)	$(\exists y\, Fy \vee P)$	(Prem)
(2)	$\quad \exists y\, Fy$	(Supp)
(3)	$\quad\quad Fa$	(Supp)
(4)	$\quad\quad (Fa \vee P)$	($\vee$I 3)
(5)	$\quad\quad \exists y(Fy \vee P)$	($\exists$I 4)
(6)	$\quad \exists y(Fy \vee P)$	($\exists$E 2, 3–5)
(7)	$\quad P$	(Supp)
(8)	$\quad (Fa \vee P)$	($\vee$I 7)
(9)	$\quad \exists y(Fy \vee P)$	($\exists$I 8)
(10)	$\exists y(Fy \vee P)$	($\vee$E 1, 2–6, 7–9)

(3) $\forall x(Fx \to P)\ \therefore\ (\exists x\, Fx \to P)$

Again, this is really very straightforward. The conclusion is a conditional, so we assume the antecedent $\exists x\, Fx$ and aim for the consequent P. And how are we going to make use of $\exists x\, Fx$? Presumably by supposing an instance is true with a view to arguing by (∃E). So let's do that:

(1)	$\mid\ \forall x(Fx \to P)$	(Prem)
(2)	$\mid\ \mid\ \exists y\, Fy$	(Supp)
(3)	$\mid\ \mid\ \mid\ Fa$	(Supp)
(4)	$\mid\ \mid\ \mid\ (Fa \to P)$	(∀E 1)
(5)	$\mid\ \mid\ \mid\ P$	(MP 3, 4)
(6)	$\mid\ \mid\ P$	(∃E 2, 3–5)
(7)	$\mid\ (\exists y Fy \to P)$	(CP 2–6)

(4) $\forall x(P \lor Fx)\ \therefore\ (P \lor \forall x\, Fx)$

It is tempting to think like this:

> From $\forall x(P \lor Fx)$ we can infer $(P \lor Fa)$.
>
> Now argue by cases. From P we can infer $(P \lor \forall x\, Fx)$.
>
> From Fa we can infer $\forall x Fx$ because a is arbitrary; and so then we can again infer $(P \lor \forall x\, Fx)$.
>
> Either way, or conclusion follows, so we are done.

Putting that formally we would get the following (supposed) derivation:

(1)	$\mid\ \forall x(P \lor Fx)$	(Prem)
(2)	$\mid\ (P \lor Fa)$	(∀E 1)
(3)	$\mid\ \mid\ P$	(Supp)
(4)	$\mid\ \mid\ (P \lor \forall x\, Fx)$	(∨I 3)
(5)	$\mid\ \mid\ Fa$	(Supp)
(6)	$\mid\ \mid\ \forall x Fx$	(∀I 5)
(7)	$\mid\ \mid\ (P \lor \forall x\, Fx)$	(∨I 6)
(8)	$\mid\ (P \lor \forall x\, Fx)$	(∨E 2, 3–4, 5–7)

Neat.

But wrong! The proof is fallacious. Look carefully. The application of ∀I at line (6) doesn't conform to the stated rule. Here is the rule again:

> We can derive $\forall\xi\alpha(\xi)$, given an available instance $\alpha(\delta)$ – so long as (i) the dummy name δ doesn't occur in any live assumption for that instance, and (ii) δ doesn't occur in the conclusion $\forall\xi\alpha(\xi)$.

But in this case, the dummy name a which we are trying to quantify on at line (6) does appear in a live assumption for that instance, namely the line (5) itself. So we can't legitimately apply the rule.

You might think that our rule is over-pernickety. Surely, you might protest, a was introduced here as an instance of a universal generalisation, so why – at least in this sort of case – don't we allow ourselves to generalize on it?

Well, think about the following example. Plainly we *can't* validity argue from the premiss $\forall x(Fx \lor \neg Fx)$ which is a logical truth, to the conclusion $(\forall x Fx \lor \forall x \neg Fx)$ which is usually false. (Example: from the logical truth that every positive integer is either even or not even, we can't conclude that either every positive integer is even or every positive integer is non-even.) But if we allowed the sort of inference move we used in our fallacious proof above, we would we able to argue like this:

(1)	$\forall x(Fx \lor \neg Fx)$	(Prem)
(2)	$(Fa \lor \neg Fa)$	($\forall$E 1)
(3)	$\quad Fa$	(Supp)
(4)	$\quad \forall x Fx$	($\forall$I 3 [? after all, a was arbitrary!])
(5)	$\quad (\forall x Fx \lor \forall x \neg Fx)$	($\lor$I 4)
(6)	$\quad \neg Fx$	(Supp)
(7)	$\quad \forall x \neg Fx$	($\forall$I 6 [? after all, a was arbitrary!])
(8)	$\quad (\forall x Fx \lor \forall x \neg Fx)$	($\lor$I 7)
(9)	$(\forall x Fx \lor \forall x \neg Fx)$	($\lor$E 2, 3–5, 6–8)

This so-called proof is a disaster. The culprit is our double offence against the constraints on using ($\forall$I), flouting the rules at lines (4) and (7).

So that leaves us with the question, how *can* we warrant the evidently correct inference $\forall x(P \lor Fx) \ \therefore\ (P \lor \forall x\, Fx)$ using our rules?

Well, start the proof in the same way, but then use that familiar dodge of assuming the negation of the desired conclusion and aiming for a reductio:

(1)	$\forall x(P \lor Fx)$	(Prem)
(2)	$(P \lor Fa)$	($\forall$E 1)
(3)	$\quad \neg(P \lor \forall x\, Fx)$	(Supp)
(4)	$\quad \neg P$	(PL – from 3)
(5)	$\quad Fa$	(PL – disjunctive syllogism from 7 and 2)
(6)	$\quad \forall x Fx$	($\forall$I 8, legitimate because a is in no assumption)
(7)	$\quad (P \lor \forall x\, Fx)$	($\lor$I 9)
(8)	$\quad \bot$	(Abs 10, 3)
(9)	$\neg\neg(P \lor \forall x\, Fx)$	(RAA 3–11)
(10)	$(P \lor \forall x\, Fx)$	(DN 12)

So, to make this into a proof by our basic rules, we'd just have to use the usual routines to flesh out the PL moves. I'll leave that to you!

(5) $(P \to \exists x\, Fx) \ \therefore\ \exists x(P \to Fx)$

This fourth example is trickier. One option would be to invoke $(P \lor \neg P)$ and then argue by cases. First case, P and the premiss gives $\exists x Fx$ and then we can argue to the desired conclusion using an ($\exists$E) argument when we remember that e.g. Fa implies $(P \to Fa)$ by propositional reasoning. Second case, $\neg P$ implies $(P \to Fa)$ by propositional reasoning and again we get our conclusion.

But that's not very pretty! Let's see if we can give an alternative derivation using just the negation and conditional rules. OK: what can now we do but take the premiss and suppose the negation of the conclusion and aim for absurdity. How can we use the conditional? Well we can suppose in addition that the antecedent is true and derive the consequent:

$(P \to \exists x\, Fx)$	(Prem)
$\quad \neg\exists x(P \to Fx)$	(Supp)
$\quad\quad P$	(Supp)
$\quad\quad \exists x\, Fx$	(MP)
$\quad\quad \vdots$	

But now what? We have no useful move to make with the wffs already in play (why not?). So we are going to need to make *another* supposition. And the obvious one to make is Fa

[of course, the particular choice of dummy name doesn't matter!] with a view to setting up a ($\exists$E) argument using the existential wff on the fourth line .....

│ $(\mathsf{P} \to \exists\mathsf{x}\,\mathsf{Fx})$	(Prem)
│ │ $\neg\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	(Supp)
│ │ │ P	(Supp)
│ │ │ $\exists\mathsf{x}\,\mathsf{Fx}$	(MP)
│ │ │ │ Fa	(Supp)
│ │ │ │ $\vdots$	

Now what to do? Let's look for a contradiction! Well, by propositional reasoning Fa gives us $(\mathsf{P} \to \mathsf{Fa})$ – a [material] conditional is true if its consequent is true – and if we existentially quantify that we can a contradiction with the first of our suppositions. Hooray!

│ $(\mathsf{P} \to \exists\mathsf{x}\,\mathsf{Fx})$	(Prem)
│ │ $\neg\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	(Supp)
│ │ │ P	(Supp)
│ │ │ $\exists\mathsf{x}\,\mathsf{Fx}$	(MP)
│ │ │ │ Fa	(Supp)
│ │ │ │ │ P	(Supp)
│ │ │ │ │ Fa	(Iter)
│ │ │ │ $(\mathsf{P} \to \mathsf{Fa})$	(CP)
│ │ │ │ $\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	($\exists$I)
│ │ │ │ $\bot$	(Abs)
│ │ │ $\bot$	($\exists$E from fourth line and the ensuing subproof)
│ │ │ $\vdots$	

So now where? We could use reductio to derive $\neg\mathsf{P}$; but how would that help? By propositional reasoning we can derive $(\mathsf{P} \to \mathsf{Fa})$ again, so $\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$, and so contradiction again. Good! But we can be brisker:

(1)	│ $(\mathsf{P} \to \exists\mathsf{x}\,\mathsf{Fx})$	(Prem)
(2)	│ │ $\neg\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	(Supp)
(3)	│ │ │ P	(Supp)
(4)	│ │ │ $\exists\mathsf{x}\,\mathsf{Fx}$	(MP 3, 1)
(5)	│ │ │ │ Fa	(Supp)
(6)	│ │ │ │ │ P	(Supp)
(7)	│ │ │ │ │ Fa	(Iter 5)
(8)	│ │ │ │ $(\mathsf{P} \to \mathsf{Fa})$	(CP 6–7)
(9)	│ │ │ │ $\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	($\exists$I 8)
(10)	│ │ │ │ $\bot$	(Abs 9, 2)
(11)	│ │ │ $\bot$	($\exists$E 4, 5–10)
(12)	│ │ │ Fa	(EFQ 11)
(13)	│ │ $(\mathsf{P} \to \mathsf{Fa})$	(CP 3–12)
(14)	│ │ $\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	($\exists$I 13)
(15)	│ │ $\bot$	(Abs 14, 2)
(16)	│ $\neg\neg\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	(RAA 2–15)
(17)	│ $\exists\mathsf{x}(\mathsf{P} \to \mathsf{Fx})$	(DN 16)

Would it have helped to use the derived rule that takes us from $\neg\exists\mathsf{x}$ to $\forall\mathsf{x}\neg$? Well, explore that!

(e*) *Do the exercise in §32.5(d) if you haven't already done it. (See the beginning of this document!) Now suppose we set up* QL*-style languages with only the universal quantifier added to the connectives. Expressions of the form $\exists\xi\alpha(\xi)$ are now introduced into such a language simply as abbreviations for corresponding expressions of the form $\neg\forall\xi\neg\alpha(\xi)$. Show that the two familiar existential quantifier rules would then be derived rules of this system.*

We need to show that

1. We can preserve ($\exists$I) in the form of the rule "We can derive the wff $\neg\forall\xi\neg\alpha(\xi)$ [$\exists\xi\alpha(\xi)$ for short] from any given instance $\alpha(\tau)$."

2. We can preserve ($\exists$E) in the form of the rule "Given $\neg\forall\xi\neg\alpha(\xi)$ [$\exists\xi\alpha(\xi)$ for short], and a subproof from the instance $\alpha(\delta)$ as supposition to the conclusion γ – where (i) the dummy name δ is new to the proof, and (ii) γ does not contain δ – then we can derive γ."

Then for (1) we just note that we can always argue in a pattern like this:

$$\begin{array}{|ll}
\alpha(\tau) & \\
\vdots & \\
\quad\begin{array}{|l} \forall\xi\neg\alpha(\xi) \\ \hline \neg\alpha(\tau) \\ \bot \end{array} & \begin{array}{l}(\text{Supp})\\(\forall\text{E})\\(\text{Abs})\end{array} \\
\neg\forall\xi\neg\alpha(\xi) & (\text{RAA})
\end{array}$$

For (2) note that we can always argue in a pattern like this:

$$\begin{array}{|ll}
\neg\forall\xi\neg\alpha(\xi) & \\
\quad\begin{array}{|ll}
\neg\gamma & \\ \hline
\quad\begin{array}{|l} \alpha(\delta) \\ \hline \vdots \\ \gamma \\ \bot \end{array} & \begin{array}{l}(\text{Supp with } \delta \text{ a new name})\\ \\ (\text{We are assuming there is such a proof from } \alpha(\delta) \text{ to } \gamma)\\(\text{Abs})\end{array} \\
\neg\alpha(\delta) & (\text{RAA}) \\
\forall\xi\neg\alpha(\xi) & (\forall\text{I, legitimate because } \delta \text{ is in no undischarged assumption}) \\
\bot & (\text{Abs})
\end{array} & \\
\neg\neg\gamma & (\text{RAA}) \\
\gamma & (\text{DN})
\end{array}$$

Alternatively, suppose we set up a QL*-style language with only a 'no' quantifier, expressed using the quantifier-former '*N*' (so* N$\xi\alpha(\xi)$ *holds when* nothing *satisfies the condition expressed by α). Give introduction and elimination rules for this quantifier. Define the universal and existential quantifier in this new language, and show how to recover their usual inference rules.*

N$\xi\alpha(\xi)$ is equivalent to $\forall\xi\neg\alpha(\xi)$. So that suggests the obvious rules for 'N':

1. (NE) From N$\xi\alpha(\xi)$, we can infer $\neg\alpha(\tau)$ for any term τ.

2. (NI) From $\neg\alpha(\delta)$ infer N$\xi\alpha(\xi)$, where δ is a dummy name not in any live assumption for $\neg\alpha(\delta)$, and δ doesn't occur in N$\xi\alpha(\xi)$.

Obviously we can define $\forall$ and $\exists$ in terms of N. Thus:

$\forall\xi\alpha(\xi)$ can be defined as abbreviating N$\xi\neg\alpha(\xi)$

$\exists\xi\alpha(\xi)$ can be defined as abbreviating $\neg$N$\xi\alpha(\xi)$

And it is then easy to check that we can then recover the familiar inference rules. Thus we have

$$\begin{array}{|ll}
\mathsf{N}\xi\neg\alpha(\xi)\ [= \forall\xi\alpha(\xi)] & \\
\neg\neg\alpha(\tau) & (\mathsf{NE}\text{, with } \neg\alpha \text{ for } \alpha) \\
\alpha(\tau) & (\text{DN})
\end{array}$$

and

$$\begin{array}{|ll}
\alpha(\delta) & (\delta \text{ not in any live assumption for this wff}) \\
\neg\neg\alpha(\delta) & (\text{PL reasoning}) \\
\mathsf{N}\xi\neg\alpha(\xi)\ [= \forall\xi\alpha(\xi)] & (\mathsf{NI}\text{, } \delta \text{ is not in this wff})
\end{array}$$

Similarly we get

$$\begin{array}{|ll}
\alpha(\tau) & \\
\quad \begin{array}{|l}\underline{\mathsf{N}\xi\alpha(\xi)}\end{array} & (\text{Supp}) \\
\quad \begin{array}{|l}\neg\alpha(\tau)\end{array} & (\mathsf{NE}) \\
\quad \begin{array}{|l}\bot\end{array} & (\text{Abs}) \\
\neg\mathsf{N}\xi\alpha(\xi)\ [= \exists\xi\alpha(\xi)] & (\text{RAA})
\end{array}$$

And further, following the answer above to the first half of this question,

$$\begin{array}{|ll}
\neg\mathsf{N}\xi\alpha(\xi)\ [= \exists\xi\alpha(\xi)] & \\
\quad \begin{array}{|l}\underline{\neg\gamma}\end{array} & (\text{Supp}) \\
\quad \begin{array}{|l}\quad \begin{array}{|l}\underline{\alpha(\delta)}\end{array}\end{array} & (\text{Supp with } \delta \text{ a new name}) \\
\quad \begin{array}{|l}\quad \begin{array}{|l}\vdots\end{array}\end{array} & \\
\quad \begin{array}{|l}\quad \begin{array}{|l}\gamma\end{array}\end{array} & (\text{We are assuming there is such a proof from } \alpha(\delta) \text{ to } \gamma) \\
\quad \begin{array}{|l}\quad \begin{array}{|l}\bot\end{array}\end{array} & (\text{Abs}) \\
\quad \begin{array}{|l}\neg\alpha(\delta)\end{array} & (\text{RAA}) \\
\quad \begin{array}{|l}\mathsf{N}\xi\alpha(\xi)\end{array} & (\mathsf{NI}\text{, legitimate because } \delta \text{ is in no undischarged assumption}) \\
\quad \begin{array}{|l}\bot\end{array} & (\text{Abs}) \\
\neg\neg\gamma & (\text{RAA}) \\
\gamma & (\text{DN})
\end{array}$$

(f*) *When discussing pairs of PL introduction and elimination rules, we saw that they fitted together in a harmonious way, with the elimination rule as it were reversing or undoing an application of the introduction rule. Can something similar be said about the pairs of QL introduction and elimination rules?*

Our earlier discussion of this idea of harmony between introduction and elimination rules was rather loose. The idea goes back to Gerhard Gentzen, the prime originator of natural deduction methods, who wrote

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol [i.e. main operator] we are dealing only 'in the sense afforded it by the introduction of that symbol'.

It isn't easy to give a precise elaboration of this idea, however. So we will continue to proceed in a rather loose, arm-waving way!

OK: The rule ($\forall$I) – in its official form – tells us that given the instance $\alpha(\delta)$ we can infer the universal quantification $\forall\xi\alpha(\xi)$ (so long as the dummy name δ doesn't appear in any live assumption on which $\alpha(\delta)$ depends).

Note, thought, that the distinction between dummy names and proper names – while a very helpful device – isn't really of the essence here. What matters is that we are taking a representative object from the domain – i.e. an object about which we are making no special assumptions – and then if we can establish that this object satisfies some condition C, we can conclude that everything satisfies C. It doesn't really matter what kind of term we use in designating the representative object and arguing about it.

So really our rule might as well have been (in many natural deduction systems it *is*) this: given the instance $\alpha(\tau)$, we can infer the universal quantification $\forall\xi\alpha(\xi)$ so long as τ doesn't appear in any live assumption.

Hence: we can think of a universal quantification $\forall\xi\alpha(\xi)$ as standardly arising by generalizing on any instance involving a term (so long as we make use of no special, distinguishing, info about that instance. Therefore arguing backwards from $\forall\xi\alpha(\xi)$ to recover its standard grounds, that will take us back from the quantified wff to any instance. Which gives what the rule ($\forall$E) says.

The rule ($\exists$I) takes us from a particular instance $\alpha(\tau)$ to the the existential quantification $\exists\xi\alpha(\xi)$. Now, that inference destroys information about which instance we start from, so we can't recover its particular grounds from a wff like $\exists\xi\alpha(\xi)$. However, suppose we have a general recipe taking us from an arbitrary satisfier of the condition expressed by α to some fixed conclusion γ. Then given $\exists\xi\alpha(\xi)$, we know it standardly comes from *some* instance, so we know we could have applied our general recipe to conclude γ. Which is in effect what ($\exists$E) says: given $\exists\xi\alpha(\xi)$ and a general recipe for getting from $\alpha(\delta)$ to the fixed conclusion γ [general in that we make no assumptions about δ, a fixed conclusion that doesn't depend on δ] we can infer γ.