Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 13: Expressive adequacy and DNF

(a) We could introduce a new *four*-place connective '$\bigsqcup$', where $\bigsqcup(\alpha, \beta, \gamma, \delta)$ is true when exactly two of $\alpha, \beta, \gamma, \delta$ are true, and is false otherwise. Show that doing this would be redundant because we can already define the new connective using the standard three connectives.

(b) More on expressive adequacy:

(1) Compare the truth tables for the down-arrow '$\downarrow$' and '$\vee$': one is formed from the other by swapping 'T's and 'F's in the last column. Define an up-arrow connective '$\uparrow$' (also symbolized '|', and then known as the 'Sheffer stroke') whose table stands in the same relation to the table for '$\wedge$'. Show that, like the down-arrow, this up-arrow connective ('NAND') taken just by itself is expressively adequate.

(2) Show that the up-arrow and down-arrow connectives are the only *binary* connectives that, taken by themselves, are expressively adequate.

(3) Define a *ternary* connective which, taken by itself, is expressively adequate.

You are supposed to spot that the answer is trivial, given what you already know. Just define e.g. $\Downarrow(\alpha, \beta, \gamma)$ so that on any line of its truth-table it's value depends on just α, β and equals $(\alpha \downarrow \beta)$ (so the third input is an idle wheel). Then $\Downarrow(\alpha, \beta, \beta)$ will do as well as $(\alpha \downarrow \beta)$ to define any truth function!

(4) Are '$\oplus$' and '$\neg$' taken together expressively adequate? What about '\$' and '$\neg$'? The tables for $\oplus$ and \$ are

α	β	$(\alpha \oplus \beta)$
T	T	F
T	F	T
F	T	T
F	F	F

α	β	γ	$\$(\alpha, \beta, \gamma)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

(c*) Assume that we are working in some PL language. Then:

(1) Show that pairs of wffs of the forms $(\alpha \wedge (\beta \vee \gamma))$ and $((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ have the same truth table.

(2) Show that pairs of wffs of the forms $((\alpha \wedge \beta) \wedge \gamma)$ and $(\alpha \wedge (\beta \wedge \gamma))$ have the same truth table. Generalize to show that any way you bracket an unmixed conjunction $\alpha \wedge \beta \wedge \gamma \wedge \ldots \wedge \lambda$ to give a properly bracketed wff expresses the same truth function. Check the comparable results for disjunctions.

(3) Show that pairs of wffs of the forms $\neg(\alpha \wedge \beta)$ and $(\neg\alpha \vee \neg\beta)$ also have the same truth tables. Generalize to show that a negated unmixed conjunction $\neg(\alpha \wedge \beta \wedge \ldots \wedge \lambda)$ has the same truth table as $(\neg\alpha \vee \neg\beta \vee \ldots \neg\lambda)$, however we insert brackets to get wffs. What are the comparable results for negated disjunctions?

(4) Say that an atom or the negation of an atom is a *basic* wff. A wff is in *disjunctive normal form* if it is, ignoring bracketing, of the form $\alpha \vee \beta \vee \ldots \vee \lambda$ for one or more disjuncts, where each disjunct is a conjunction of one or more basic wffs. Show that any wff has the same truth table as a wff in disjunctive normal form.

(5) Define an analogous notion of being in *conjunctive normal form*. Show that any wff α has the same truth table as a wff in conjunctive normal form. (Hint: consider a wff in disjunctive normal form which is equivalent to $\neg\alpha$ and take negations.)