Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 29: Simple QL translations

In the language QL_1, the *proper names* are just

- m: Socrates,
- n: Plato,
- o: Aristotle;

and the *predicates* are just

- F: ① is a philosopher,
- G: ① is a logician,
- H: ① is wise,
- L: ① loves ②,
- M: ① is a pupil of ②,
- R: ① prefers ② to ③.

The *domain of quantification* for QL_1: people, past and present.

(a) *Translate the following into* QL_1 *as best you can:*

(1) Aristotle is a pupil of Socrates and Plato.

$\simeq (\mathsf{Mom} \land \mathsf{Mon})$

Remember: the conjunction can't conjoin *names* in a QL language!

(2) Plato taught Aristotle only if Socrates taught Plato.

$\simeq (\mathsf{Mon} \to \mathsf{Mnm})$

Given the resources of QL_1, the best we can do to translate *a taught b* is to render it as we would render *b is a pupil of a* (and hope that the tense doesn't matter!).

(3) If Plato is a pupil of someone, he is a pupil of Socrates.

$\simeq (\exists \mathsf{x}\, \mathsf{Mnx} \to \mathsf{Mnm})$

Note that the quantifier is inside the antecedent of the conditional. $\exists\mathsf{x}(\mathsf{Mnx} \to \mathsf{Mnm})$ means something quite different – why?

Of course, you can use a variable other than 'x'! – we won't keep repeating this.

(4) Some philosophers are not wise.

$\simeq \exists\mathsf{x}(\mathsf{Fx} \land \neg\mathsf{Hx})$

(5) Not all philosophers are wise.

$\simeq \neg\forall\mathsf{x}(\mathsf{Fx} \to \mathsf{Hx})$

If you got either of (4) or (5) wrong, then you probably need to re-read Chapters 27–29!

But one observation. The English sentences (4) and (5) are logically equivalent – necessarily, one is true if and only if the other is. Does that mean we could have translated them the same way? No. Or at least, no, not if we require translations to reflect, as best we can, not only the overall truth-conditions of sentences but also their internal logical structure. (4) has a negation inside the scope of a 'some'; (5) has its negation outside the scope of the 'all'; we want our respective translations to reflect such different logical structures when we can – as our translations do.

However, note too how the translations also import structure which isn't in the English – a conjunction in the first case, a conditional in the second. This is the price we pay for only having *unary* quantifier constructions in QL languages as explained in §27.2.

(6) Any logician loves Aristotle.

$\simeq \forall x(Gx \rightarrow Lxo)$

(7) No one who is a wise philosopher prefers Plato to Aristotle.

$\simeq \forall x((Fx \wedge Hx) \rightarrow \neg Rxno)$
$\simeq \neg\exists x((Fx \wedge Hx) \wedge Rxno)$

As noted in 29.3, QL languages do not have a built-in 'no' quantifier, but there are always two ways of translating 'no' propositions, one using '∀' and one using '∃'.

We might wonder though about translating 'wise philosopher' as we do. Think for a moment more generally about rendering an English proposition of the form *n is (an) FG*. Sometimes, a formal version of the shape $(Fn \wedge Gn)$ is evidently correct: for example 'Jill is a Scottish philosopher' tells us, exactly, that Jill is Scottish and that she is a philosopher, and can be rendered by a formally conjunctive wff. By contrast, for example, consider 'Jack is an alleged murderer': this does not tell us that Jack is alleged and Jack is a murderer!

What about 'Jo is a poor student'? That's ambiguous. There is a reading which tells us that Jo is a student, and (rather typically for students!) Jo hasn't much money: this could of course be formally rendered by a conjunctive wff. And there is the reading which tells us that Jo isn't very good as student, which isn't baldly conjunctive in the same way.

Now to come to the present case: does 'Plato is a wise philosopher' tell us, conjunctively, that Plato is a philosopher and Plato is wise. Arguably yes, in many contexts. On the other hand, we might think that someone could be wise *as* a philosopher – yet be a foolish person: if we say 'David is a wise philosopher but otherwise quite hopeless at life' then presumably we wouldn't want to give the baldly conjunctive rendering for 'David is a wise philosopher' in this context.

But we won't fuss about this any more here. We will assume for the purposes of our examples that we can render 'wise philosopher' conjunctively into our QL language.

(8) Whoever is a pupil of Plato is wise.

$\simeq \forall x(Mxn \rightarrow Hx)$

(9) Not every wise logician is a pupil of Aristotle.

$\simeq \neg\forall x((Fx \wedge Hx) \rightarrow Mxo)$

(10) Any logician is a wise philosopher.

$\simeq \forall x(Gx \rightarrow (Fx \wedge Hx))$

(11) Aristotle prefers no philosopher to Plato.

$\simeq \forall x(Fx \rightarrow \neg Roxn)$
$\simeq \neg\exists x(Fx \wedge Roxn)$

(12) Some wise people aren't philosophers, and some aren't logicians.

$\simeq (\exists x(Hx \wedge \neg Fx) \wedge \exists y(Hy \wedge \neg Gy))$

'Some wise people aren't philosophers' can be rendered as $\exists x(Hx \wedge \neg Fx)$, and 'Some wise [people] aren't logicians' can be rendered as $\exists x(Hx \wedge \neg Gx)$; so their conjunction could quite properly be rendered

$\simeq (\exists x(Hx \wedge \neg Fx) \wedge \exists x(Hx \wedge \neg Gx))$

using the variable 'x' again for the second clause: that's a perfectly kosher wff of our language. But it is good default policy to use a new variable every time you introduce a new quantifier into a sentence, because that will prevent mishaps (quantifiers getting linked to places in predicates in unintended ways!).

Note by the way that the following will not do as a translation:

$\simeq \exists x(Hx \wedge (\neg Fx \wedge \neg Gx))$

which means something different from (12) – what?

(13) Only philosophers love Aristotle.

$\simeq \forall x(Lxo \to Fx)$

If only philosophers love Aristotle, then anyone who loves Aristotle is a philosopher – which is what our formal rendition captures. But does (13) say more? Well it certainly doesn't imply that *all* philosophers love Aristotle. Does it imply though that at least *some* philosophers do? Do we need a more complex translation, adding an existential clause? Arguably not.

Consider another example of the 'only' construction: There's a 1500 page reference book on set theory which I recommend, saying 'It's very useful, but only masochists will read this book from cover to cover'. Am I thereby claiming that someone *will* actually read it from cover to cover? In this case, surely not. A simple rendition of the form $\forall x(Mx \to Rx)$ *without* an added clause $\exists x(Mx \wedge Rx)$ seems right.

(14) Not only philosophers love Socrates.

Prefixing a 'not' to the previous translation (and changing the name) gives us

$\simeq \neg\forall x(Lxm \to Fx)$

which is equivalent (why?) to

$\simeq \exists x(Lxm \wedge \neg Fx)$

And that seems right. 'Not only philosophers love Socrates' surely is true just when there are some who love Socrates and aren't philosophers.

Given that it seems right to translate the 'not only' proposition by negating the (13)-style translation of the corresponding 'only' proposition, that seems further evidence for the correctness of our simple rendition of the 'only' proposition.

(15) Socrates is a philosopher whom everyone wise loves.

$\simeq (Fm \wedge \forall x(Hx \to Lxm))$

(16) Only some logicians love Plato.

There is a difference between the 'only' and 'only some' quantifiers, a difference between (13) and (16). This present proposition means some logicians love Plato but not all do, so can be rendered

$\simeq (\exists x(Gx \wedge Lxm) \wedge \neg\forall y(Gy \to Lxm))$

(17) No philosophers or logicians are wise.

$\simeq \forall x((Fx \vee Gx) \to \neg Hx)$
$\simeq \neg\exists x((Fx \vee Gx) \wedge \neg Hx)$

(18) All philosophers and logicians are wise.

$\simeq (\forall x(Fx \to Hx) \wedge \forall y(Gx \to Hx))$
$\simeq \forall x((Fx \vee Gx) \to Hx)$

But *not*

$\simeq \forall x(Fx \wedge Gx) \to Hx)$

(18) doesn't say that you have to be both a philosopher and a logician to be wise! That's why the third proposed rendition won't do. The claim is the everyone who is a philosopher is wise and so too is everyone who is a logician (the first rendition). Or equivalently: take anyone who is a philosopher or a logician, then (either way) they'll be wise (the second rendition).

(b) [Perhaps this question should have been starred and/or cut down!] *Which of these pairs of wffs are equivalent, which not, and why?*

(1) $(\exists x Fx \lor \exists x Gx)$, $\exists x(Fx \lor Gx)$

Evidently equivalent. Each disjunct of (1_L) [the left-hand wff] implies (1_R) [the right-hand wff]. Hence overall (1_L) implies (1_R). Conversely, given (1_R) we can pick some object which satisfies either F or G; in the first case this object witnesses that the first disjunct of (1_L) is true, in the second case this object witnesses that the second disjunct of (1_L) is true; so either way (1_L) is true. Hence (1_R) implies (1_L).

(2) $(\exists x Fx \land \exists x Gx)$, $\exists x(Fx \land Gx)$

Not equivalent. (2_R) [the right-hand wff] entails (2_L) [the left-hand wff]. But not vice versa. Suppose we interpret F as 'very young', G as 'very old', with the domain people. That makes (2_L) true and (2_R) false.

(3) $(\exists x Fx \land \exists x Gx)$, $\exists x(Fx \land \exists y Gy)$

Equivalent. (3_L) is trivially equivalent to $(\exists x Fx \land \exists y Gy)$ by relabelling of variables. [Why is this legitimate? because (3_L) is the conjunction of two 'non-interacting' components $\exists x Fx$ and $\exists x Gx$, and we can systematically change the variable in either component.]

And if α doesn't involve the variable x, $(\exists x Fx \land \alpha)$ is equivalent to $\exists x(Fx \land \alpha)$. [Why? See §29.6. Or, from first principles, reflect $(\exists x Fx \land \alpha)$ is true if both components are true, and hence just α is true and there is some way of choosing a reference for n which makes Fn true, and hence just if $\exists x(Fx \land \alpha)$ is true.]

So (3_L) is equivalent to $(\exists x Fx \land \exists y Gy)$ which is is equivalent to to (3_R).

(4) $(\exists x Fx \land \exists x Gx)$, $\exists x \exists y(Fx \land Gy)$

Equivalent by a similar argument as for (3).

(5) $\exists x \forall y(Fx \land Gy)$, $\forall y \exists x(Fx \land Gy)$

Something of the form $\exists x \forall y \alpha$ evidently implies the corresponding $\forall y \exists x \alpha$ (why?).

But you know that in general we can't argue in the other direction from something of the form $\forall y \exists x \alpha$ to the corresponding $\exists x \forall y \alpha$ – famously, ∀y∃x x loves y does not imply ∃x∀y x loves y.

But in *this* special case we can validly argue not only from (5_L) to (5_R), but also from (5_R) to (5_L) – the essential point being that there is here no real connection between what the two quantifiers are quantifying (they are, so to speak, working quite independently so it doesn't matter which order we consider them in).

If that's not clear, it might help to consider a miniature domain (often an illuminating trick!). So consider a *two* object domain, containing just the two objects named by m and n. So contrast these cases:

1. If $\forall y \exists x Lxy$ is true, that requires *both* (i) $\exists x Lxm$ *and* (ii) $\exists x Lxn$. (i) requires either Lmm or Lnm; and (ii) requires either Lmn or Lnn. But these are independent requirements; we could e.g. make $\forall y \exists x Lxy$ true by having (i) just Lmm true and (ii) just Lnn true. And then $\exists x \forall y Lxy$ will be false.
2. If $\forall y \exists x(Fx \land Gy)$ is true, that requires *both* (i′) $\exists x(Fx \land Gm)$ *and* (ii′) $\exists x(Fx \land Gn)$. So that means we must have both Gm and Gn. But also, to make (i′) true overall, we must have either Fm or Fn. But whichever it is will *also* be enough to make (ii′) true overall. Suppose e.g. it is Fm. Then we'll have both $(Fm \land Gm)$ and $(Fm \land Gn)$, and hence $\forall y(Fm \land Gy)$ and hence $\exists x \forall y(Fx \land Gy)$.

(6) $\exists x \forall y(Fx \lor Gy)$, $\forall x \exists y(Fy \lor Gx)$

Change the variables in (6_L) to get the equivalent $\exists y \forall x(Fy \lor Gx)$. Then that in turn is equivalent to (6_R) by a similar argument as in (5).

(7) $(\forall x Fx \to \forall x Gx)$, $\exists x(Fx \to \forall y Gy)$

Change the variables in (7_L) to get the equivalent $(\forall x Fx \to \forall y Gy)$. Then we can export the first quantifier in the antecedent of the conditional to get the equivalent $\exists x(\forall x Fx \to \forall y Gy)$. Why?

We explained this in 29.6. Recall, $(\forall x Fx \to \forall y Gy)$ is equivalent to $(\neg\forall x Fx \lor \forall y Gy)$ is equivalent to $(\exists x \neg Fx \lor \forall y Gy)$ is equivalent to $\exists x(\neg Fx \lor \forall y Gy)$ is equivalent to $\exists x(Fx \to \forall y Gy)$.

(8) $(\forall x Fx \to \forall x Gx)$, $\exists x \forall y(Fx \to Gy)$

Things are getting trickier!. But we know from (7) that $(\forall x Fx \to \forall x Gx)$ is equivalent to $\exists x(Fx \to \forall y Gy)$. So the question now becomes whether the latter is equivalent to $\exists x \forall y(Fx \to Gy)$.

Well, note that $(Fn \to \forall y Gy)$ and $\forall y(Fn \to Gy)$ are straightforwardly equivalent (why?).

So to say that there is *something* which has the property n has when $(Fn \to \forall y Gy)$ must be equivalent to saying that there is *something* which has the property n has when $\forall y(Fn \to Gy)$. But that tells us that, indeed, $\exists x(Fx \to \forall y Gy)$ is equivalent to $\exists x \forall y(Fx \to Gy)$.

(9) $(\forall x Fx \to \forall x Gx)$, $\forall y \exists x(Fx \to Gy)$

(8_R) is in fact equivalent to (9_R) – compare the observation in the answer to (5) on cases where we can swap initial quantifiers. Hence (9_L) (which is (8_L) again) is indeed equivalent to (9_R). [That (8_R) implies (9_R) is trivial (why?): we'll give a formal proof that (9_R) implies (8_R) in Exercises 33.]

(c*) *Use equivalences you now know about to outline a proof that every wff is equivalent to one in* prenex *form, where all quantifiers are at the beginning of the wff.*

A challenging question at this stage (and would be difficult to answer in a technically rigorous way with the resources you currently have).

Suppose first that a given wff is built up using just quantifiers, conjunctions and disjunctions: and suppose that no variable is used in more than one quantifier (so we get no 'clash of variables' as we move quantifiers around). Then it's easy! We know we can dumbly pull a quantifier outside a conjunction or disjunction using equivalences like that between $(\exists x Fx \land \alpha)$ and $\exists x(Fx \land \alpha)$ and that between $(\alpha \lor \forall x Gx)$ and $\forall x(\alpha \lor Gx)$ – when α doesn't contain occurrences of x. We can keep dragging quantifiers further and further to the left [though not letting one quantifier 'overtake' another – their relative order stays the same!]

So now we need to beef up this plan to accommodate negation signs and conditionals (for quantifiers *do* actively interact with these, as we know). But that's not too difficult. Here's our plan then. Given a wff with quantifiers in it and perhaps negations and/or conditionals:

1. Pre-process it by relabelling variables to making sure none are used by more than one quantifier [so transform the likes of $(\forall x Fx \land \exists x Gx)$ into $(\forall x Fx \land \exists y Gy)$, etc. etc.]

2. Pre-process another stage by replacing those pesky conditionals with disjunctions and negations.

3. Then push negation signs *to the right* as far as they will go [flipping quantifiers as a negation passes, using De Morgan's Laws to deal with a negated conjunction or negated disjunction, eliminating any double negations]. This gets negations out of harms way, not interfering with other operators, with any remaining negation signs attaching directly to predicates.]

4. We now have an equivalent wff which is built from (possibly negated) predicates-attached-to-names/variables, using just quantifiers, conjunction and disjunction. So as we said (starting work from the first quantifier) we can pull any quantifiers inside conjunctions or disjunctions *to the left*, to the front of the wff, using the sort equivalences we've noted. And we are done!

Except that we've cheated! For this to work, we have to assume that the various equivalences we've noted continue to work when applied to inner *parts* of wffs as well as whole wffs [for example we need might need $(\exists x Fx \wedge Gy)$ to be equivalent to $\exists x(Fx \wedge Gy)$, even though that y is dangling free]. This is where we are arm-waving and being not quite rigorous! – but the assumption hopefully looks safe enough, and we should be able to cash it out!

For a little more on this and some examples, see e.g. these slides.

(d*) *We can render 'Plato and Aristotle are philosophers' by e.g.* $(Fn \wedge Fo)$. *Why can't we render 'Plato and Aristotle are classmates' by e.g.* $(Cn \wedge Co)$*? What does this sort of case tell us about some expressive limitations of* QL *languages?*

An (interpreted) wff of the form $(Cn \wedge Co)$ entails the separate claims Cm and Co. But from 'Plato and Aristotle are classmates' we can't infer 'Plato is a classmate' (which hardly makes standalone sense). Similarly from 'Plato and Aristotle met in the pub' we can't infer 'Plato met in the pub' (which doesn't make standalone sense either).

Let's say that a predicate F is *distributive* if *m and n are F* entails *m is F* and *n is F*, and the predicate is *collective* otherwise. Then an initial observation is that our QL languages cope with *distributive* predicates and not collective ones like 'are classmates', 'met in the pub'.

And don't think that collective predicates are a minor oddity of informal ordinary language that we logicians can cheerfully just ignore when treating serious claims that might appear e.g. in maths (or when discussing logic!). Consider e.g. this claim about some points in the plane: *A and B and C are collinear*: we obviously can't treat this as a bald conjunction of *A is collinear* and *B is collinear* and *C is collinear*! Consider too the predicates 'are parallel', 'are isomorphic', and the logical predicate 'are inconsistent'.

Predicates like 'are classmates', 'met in the pub', 'are collinear', 'are inconsistent' take *plural* subjects – these might be list-like terms like 'Plato and Aristotle' or 'the points A and B and C', but also plural descriptions like 'the members of the logic class', 'the Brontë sisters', 'the premisses' and plural quantifiers like 'some women', 'all the soldiers'. QL languages don't have direct counterparts to these expressions. And, to quote Alex Oliver and Timothy Smiley "It is no good dismissing grammatical number as a logically irrelevant complication like person or gender, since plural expressions are crucially involved in valid patterns of argument. To take an elementary example, 'The Bront sisters supported one another; the Bront sisters were Anne, Charlotte and Emily; so Anne, Charlotte and Emily supported one another'."

So a formal logic based on QL languages can only take us so far. For an indication of what a logical language which can deal with plurals might look like, see for a start this introductory encyclopaedia article.