Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 38: Relations and Leibniz's Law

(a) *Check you know what it is for a relation to be transitive, symmetric and reflexive. In addition*

(i) A relation R *defined over a given domain is Euclidean just if, whenever* a *has* R *to* b *and* a *has* R *to* c*, then* b *has* R *to* c*;*

(ii) R *is asymmetric if, whenever* a *has* R *to* b*, then* b *does not have* R *to* a*;*

(iii) R *is irreflexive if no object has* R *to itself.*

Give examples of relations which are neither reflexive nor irreflexive, and neither symmetric nor asymmetric.

The relation *admires* is not reflexive (not everyone admires themself), and is not irreflexive (it isn't ruled out that a person admires themselves), is not symmetric (if Romeo admires Juliet it doesn't follow that Juliet admires Romeo) and is not asymmetric (mutual admiration isn't impossible).

Which of the following are true? Give QL *proofs for examples of the true claims.*

(1) *If* R *is asymmetric, it is irreflexive.*

True. If R is asymmetric, then for any a, b, if a has R to b, then b doesn't have R to a. So, as a special case, if a has R to a, then a doesn't have R to a – whence a can't be R to a.

Formally, we can show the inference $\forall x \forall y(Rxy \to \neg Ryx) \therefore \forall x \neg Rxx$ is valid.

(1)	$\forall x\forall y(Rxy \to \neg Ryx)$	(Prem)
(2)	$\forall y(Ray \to \neg Rax)$	(∀E I)
(3)	$(Raa \to \neg Raa)$	(∀E 2)
(4)	$\quad Raa$	(Supp)
(5)	$\quad \neg Raa$	(MP 4, 3)
(6)	$\quad \bot$	(Abs 4, 5)
(7)	$\neg Raa$	(RAA 4–6)
(8)	$\forall x \neg Rxx$	(∀I 7)

(2) *If* R *is transitive and irreflexive, it is asymmetric.*

True. Suppose R is (i) transitive and (ii) irreflexive but (iii) *not* asymmetric. Then by (iii), for some a, b, both a has R to b and b doesn't have R to a. So, by (i) a has R to a. Which contradicts (ii). So if (i) and (ii), then not (iii) – i.e. R is asymmetric.

Formally, we need to validate $\forall x\forall y\forall z((Rxy \land Ryz) \to Rxz), \forall x \neg Rxx \therefore \forall x\forall y(Rxy \to \neg Ryx)$.

(1)	$\forall x\forall y\forall z((Rxy \land Ryz) \to Rxz)$	(Prem)
(2)	$\forall x \neg Rxx$	(Prem) [Now aim for $(Rab \to \neg Rba)$ and generalize!]
(3)	$\quad Rab$	(Supp) [Set off on a conditional proof!]
(4)	$\quad\quad Rba$	(Supp) [We need to show $\neg Rba$; try reductio!]
(5)	$\quad\quad \forall y\forall z((Ray \land Ryz) \to Raz)$	(∀E 1)
(6)	$\quad\quad \forall z((Rab \land Rbz) \to Raz)$	(∀E 5)
(7)	$\quad\quad ((Rab \land Rba) \to Raa)$	(∀E 6)
(8)	$\quad\quad (Rab \land Rba)$	(∧I 3, 4)
(9)	$\quad\quad Raa$	(MP 8, 7)
(10)	$\quad\quad \neg Raa$	(∀E 2)
(11)	$\quad\quad \bot$	(Abs 9, 10)
(12)	$\quad \neg Rba$	(RAA 4–6)
(13)	$(Rab \to \neg Rba)$	(CP 3–12)
(14)	$\forall y(Ray \to \neg Rya)$	(∀I 13)
(15)	$\forall x\forall y(Rxy \to \neg Ryx)$	(∀I 14)

(3) *If* R *is transitive and symmetric, it is reflexive*

We have just shown that if R is transitive and irreflexive, it is asymmetric. That's equivalent to showing that if R is transitive and not asymmetric, it is not irreflexive (why?). So have we established (3).

No. Because *not asymmetric* does not mean *symmetric* and *not irreflexive* does not mean *reflexive* (why?).

And in fact (3) is false. For an extreme case, take a R that applies to nothing in the relevant domain. It is transitive (trivially, in any case where a is R to b, and b is R to c, a is R to c); it is symmetric (trivially, in any case where a is R to b, then b is R to ca); but it is not reflexive (indeed, it is never the case that a is R to a).

For a less extreme case, take the relation 'is the same cat as' defined over all mammals; it is transitive and symmetric, but not reflexive (a dog is not the same cat as itself!)

A moral of this example: the three components in the definition of an equivalence relation are all essential – we can't just drop the reflexivity requirement as redundant.

(4) *If* R *is an equivalence relation, it is Euclidean.*

Remember, an equivalence relation is transitive, symmetric and reflexive. Suppose (i) a is R to b, and (ii) a is R to c. By symmetry, (iii) b is R to a. By transitivity, from (iii) and (ii), (iv) b is R to c. Which is what we needed to show.

Note that we didn't need to appeal to R's reflexivity. So formally, it will be enough to validate $\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz), \forall x\forall y(Rxy \rightarrow Ryx) \therefore \forall x\forall y\forall z((Rxy \wedge Rxz) \rightarrow Ryz)$.

(1)	$\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)$	(Prem)
(2)	$\forall x\forall y(Rxy \rightarrow Ryx)$	(Prem)
(3)	$(Rab \wedge Rac)$	(Supp) [Why is this a natural supposition to make?]
(4)	Rab	($\wedge$E 3)
(5)	Rac	($\wedge$E 3)
(6)	$\forall y(Ray \rightarrow Rya)$	($\forall$E 2)
(7)	$(Rab \rightarrow Rba)$	($\forall$E 6)
(8)	Rba	(MP 4, 7)
(9)	$(Rba \wedge Rac)$	($\wedge$I 8, 5)
(10)	$\forall y\forall z((Rby \wedge Ryz) \rightarrow Rbz)$	($\forall$E 1)
(11)	$\forall z((Rba \wedge Raz) \rightarrow Rbz)$	($\forall$E 10)
(12)	$((Rba \wedge Rac) \rightarrow Rbc)$	($\forall$E 11)
(13)	Rbc	(MP 3, 12)
(14)	$((Rab \wedge Rac) \rightarrow Rbc)$	(CP 3–13)
(15)	$\forall z((Rab \wedge Raz) \rightarrow Rbz)$	($\forall$I 14)
(16)	$\forall y\forall z((Ray \wedge Raz) \rightarrow Ryz)$	($\forall$I 15)
(17)	$\forall x\forall y\forall z((Rxy \wedge Rxz) \rightarrow Ryz)$	($\forall$I 16)

(5) *If* R *is asymmetric and Euclidean, it is irreflexive.*

Are you awake? We've just seen that if R is asymmetric, it is irreflexive. So of course, if it is is asymmetric and Euclidean it is still irreflexive!

(6) *If* R *is Euclidean and reflexive, it is an equivalence relation.*

Assume R is Euclidean and reflexive. Suppose a is R to b. Then, since R is reflexive we also have a is R to a, so by the Euclidean property, b is R to a. So R is symmetric.

Assume R is Euclidean and reflexive and hence symmetric. Suppose (i) a is R to b and also (ii) b is R to c. By symmetry, (iii) b is R to a, and so by the Euclidean property, from (iii) and (ii) it follows that (iv) a is R to c. So if (i) and (ii), then (iv) – which gives us transitivity.

So a Euclidean and reflexive relation is transitive, symmetric as well as reflexive – i.e. is an equivalence relation.

The formal proofs of symmetry and transitivity can follow the informal sketches exactly. So let's leave them to look after themselves! They will look very similar to the formal proofs above, and will involve no significantly new ideas.

Take the domain containing just the five numbers $0, 1, 2, 3, 4$. *How many different equivalence relations can be defined over the domain? (For this purpose, count relations as the same if they have the same extensions.)*

We know that an equivalence relation partitions its domain into (non empty) equivalence classes. So we are in fact asking: how many ways are there of partitioning the set $\{0, 1, 2, 3, 4\}$. The answer is in fact 52. It doesn't matter two hoots if you got the counting wrong – as long as you spotted that the neat way to answer the question is to count partitions.

How to get the answer, by brute force:

i. There's ONE case where all those elements are in the same single partition.

ii. There are FIVE cases where four elements are in one partition, and a stray singleton is in another.

iii. There are TEN cases where three elements are in one partition, and two are in another. [Just remember there are ten ways of choosing two things from five/so ten ways of choosing three things from five!]

iv. There are TEN cases where three elements are in one partition, and with the other two in two separate partitions.

v. There are FIFTEEN cases were the elements are partitioned into a set of two, another set of two, and a remaining one. [There are five ways of choosing the singleton, and then three ways of splitting the remaining four items into two pairs.]

vi. There are TEN cases were the elements are partitioned into a set of two, and three singletons.

vii. There is ONE remaining case where the elements are partitioned into five singleton sets

So that makes 52 ways of partioning 5 things. If you want a more systematic story, find out about Bell numbers.

(b) *In §38.4 we stated the following principle:*

(LL′) Suppose that in the sentence $C(n)$, *the context* C *attributes some property to the object referred to by the term* n *(the same property whichever term* n *completes the sentence). Then if* a *and* b *are co-referential terms and* $C(a)$ *is true, then* $C(b)$ *is also true.*

This is quite understandable as it is. But give a version which might satisfy a logician who is very *pernickety about the use of quotation marks.*

A pause for a reality check. What, of anything, is wrong with (LL′)?

In fact, even a pernickety reader should allow that (LL′) is in good order without any additional quotation marks, so long as the right conventions governing the use of the schematic expressions 'C', 'n', '$C(n)$', etc., are in place.

Suppose 'n' stands in for some term, and '$C(n)$' stands in for a sentence involving one or more occurrences of that term. Then of course '$C(a)$' and '$C(b)$' stand in for the two sentences which we get by substituting the terms a and b respectively for occurrences of n. Now remove those terms from the sentences: we are left with a common context: we could use some 'gap-marking' notation indicating the slots where the terms would go, but there is no harm in simply using 'C' to indicate the result of removing occurrences of the term n from the sentence $C(n)$.

With all that understood as pre-amble, we don't need further quotation conventions for (LL′) to be in good order.

(c*) *Which, if any, of the following arguments involving identity claims are valid? How do they relate to Leibniz's Law?*

(1) *Tubby is so-called because of his size. Tubby is none other than Dr Jones. Hence, Dr Jones is so-called because of his size.*

Evidently not a valid argument! How does this square with Leibniz's Law?

Well, what does the context C '... is so-called because of his size' say of Tubby in the first premiss? It says, truly, that he is called 'Tubby' because of his size. What does the context C say of Dr Jones in the conclusion? It says, falsely, that he is called 'Dr Jones' because of his size. So note that this context C does *not* attribute the *same* property to the object referred to by the name n in the sentence $C(n)$, irrespective of which name we use. So the principle (LL$'$) doesn't apply.

Of course, there is a perfectly valid argument which uses Leibniz's Law in the vicinity:

> Tubby is called 'Tubby' because of his size. Tubby is none other than Dr Jones. Hence, Dr Jones is called 'Tubby' because of his size.

The first premiss is equivalent to the first premiss of (1). But the conclusion, which must be true if the premisses are true, is different to the conclusion of (1).

(2) *Few people have heard of Besarionis dze Jughashvili. Jughashvili is in fact Stalin. Therefore few people have heard of Stalin.*

People will probably divide over this (and I haven't done a big enough survey to say where majority opinion lies). I wouldn't want to insist that there is a right answer!

Some (a minority?) will say that this a valid argument whose first premiss is actually false – *lots* of people, they will say, have heard of Jughashvili, just not under that name.

Some (a majority?) will say that the premisses of this argument are both true and its conclusion is false, so the argument is invalid. How does this square with Leibniz's Law? Presumably, if you take the premiss as true, you are reading it as saying that few people have heard of Besarionis dze Jughashvili as such, or have heard of him under the name 'Besarionis dze Jughashvili'. From that, and the truth Jughashvili is Stalin, we get another truth by Leibniz's law, few people have heard of Stalin under the name 'Besarionis dze Jughashvili'. But we can't infer that few people have heard of Stalin under the name 'Stalin'.

(3) *George Orwell is a well-known author. George Orwell is Eric Blair. So Eric Blair is a well-known author.*

A similar case perhaps, though I could imagine people jumping different ways on this question and the previous one.

The following is surely a straightforwardly valid argument by Leibniz's Law: "George Orwell is an author. George Orwell is Eric Blair. So Eric Blair is an author." If George has written a book or two, and Eric is George, then Eric has written a book or two. Does inserting "well-known" make a difference? Some would say not – Eric Blair is a well-known author, just not under that name. Some however might read "George Orwell is a well-known author" as "George Orwell is an author well known under that name", and then the argument will fail.

(4) *Necessarily, nine is nine. The number of planets is nine. So, necessarily, the number of planets is nine.*

This is a tease, which raises a couple of issues about the modal notion of necessity and about the handling of so-called definite descriptions like 'the number of planets'. I'll leave the issues hanging, for you to think about ......