Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 16: More about tautological entailment

(a*) Our book definition says that $\alpha_1, \alpha_2, \ldots, \alpha_n \vDash \gamma$ if and only if there is no valuation of the atoms *involved in the relevant wffs* which makes the αs all true and γ false. Show that we could equivalently have said: The wffs $\alpha_1, \alpha_2, \ldots, \alpha_n \vDash \gamma$ if and only if there is no valuation of *all the language's atoms* which makes the αs all true and γ false. (Hint: use the fact that the values of atoms that don't appear in a wff can't affect the value of that wff.)

(b*) Why are the following true? (Hint: make use of the previous exercise.)

(1) Any two tautologies are tautologically equivalent.

(2) If $\alpha \vDash \gamma$, then $\alpha, \beta \vDash \gamma$.

(3) If $\alpha, \beta \vDash \gamma$ and β is a tautology, then $\alpha \vDash \gamma$.

(4) If $\alpha \vDash \beta$ and $\beta \vDash \gamma$, then $\alpha \vDash \gamma$.

(5) Suppose $\beta \approx \beta'$ (i.e. $\beta \vDash \beta'$ and $\beta' \vDash \beta$). Then for any wffs α, γ both (i) $\alpha \vDash \beta$ if and only if $\alpha \vDash \beta'$, and (ii) $\beta \vDash \gamma$ if and only if $\beta' \vDash \gamma$.

(6) Replacing a subformula of a wff by an equivalent expression results in a new wff equivalent to the original one.

(c*) Some new notation. Alongside the use of lower-case Greek letters for individual wffs, it is common to use upper-case Greek letters such as 'Γ' (*Gamma*) and 'Δ' (*Delta*) to stand in for some wffs – zero, one, or many.

Further, we use 'Γ, α' for the wffs Γ together with α. We also use '$\Gamma \cup \Delta$' for the wffs Γ together with the wffs Δ.

Now prove these generalized versions of the some of the claims in (b):

(1) If $\Gamma \vDash \gamma$, then $\Gamma, \alpha \vDash \gamma$.

(2) If $\Gamma, \alpha \vDash \gamma$ and α is a tautology, then $\Gamma \vDash \gamma$.

(3) If $\Gamma \vDash \beta$ and $\Delta, \beta \vDash \gamma$, then $\Gamma \cup \Delta \vDash \gamma$.

(4) State and prove a general version of (b5) which allows for inferences with multiple premisses.

We will normally be interested in cases where we are dealing with only finitely many wffs Γ. But would (1) to (3) still be true if Γ and/or Δ were infinitely many?

(d*) Given the results

(1′) $(\alpha \wedge \beta) \approx (\beta \wedge \alpha)$
(2′) $(\alpha \wedge (\beta \wedge \gamma)) \approx ((\alpha \wedge \beta) \wedge \gamma)$
(3′) $(\alpha \wedge (\beta \vee \gamma)) \approx ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

it can be said 'conjunction is commutative', 'conjunction is associative', 'conjunction distributes over disjunction'. Investigate and explain. What parallel claims apply to disjunction?

(e*) Find out more about the idea of 'working backwards' (touched on in Ch. 16) by looking at the online supplement on propositional truth trees.