Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 16: More about tautological entailment

(a*) *Our book definition says that* $\alpha_1, \alpha_2, \ldots, \alpha_n \vDash \gamma$ *if and only if there is no valuation of the atoms* involved in the relevant wffs *which makes the* α*s all true and* γ *false. Show that we could equivalently have said: The wffs* $\alpha_1, \alpha_2, \ldots, \alpha_n \vDash \gamma$ *if and only if there is no valuation of* all the language's atoms *which makes the* α*s all true and* γ *false. (Hint: use the fact that the values of atoms that don't appear in a wff can't affect the value of that wff.)*

It is trivial that if (A) there is no valuation of *all the language's atoms* which makes the αs all true and γ false, then in particular (B) there is no valuation of the atoms *involved in these wffs* which makes the αs all true and γ false.

So what we need to show is that, conversely, if (B), then (A) – or equivalently, if not-(A) then not-(B). Well, suppose not-(A), i.e. suppose there *is* a valuation of *all the language's atoms* which makes the αs all true and γ false. But we know that the values of atoms that don't appear in a wff can't affect the value of that wff (remember, we determine the value of a wff on a valuation of atoms by up through the wff's parse tree, and only atoms in the wff will appear in the parse tree and so only the valuation of those atoms are relevant). So given there is a valuation of all the language's atoms which makes the αs all true and γ false, and it is only the values of the atoms actually occurring in the αs and γ that can determine their values, it follows that there is a valuation of *those* atoms which makes the αs all true and γ false, giving us not-(B).

Since (A) and (B) are therefore equivalent, it makes no difference which we use in the definition of tautological entailment.

(b*) *Why are the following true? (Hint: make use of the previous exercise.)*

(1) *Any two tautologies are tautologically equivalent.*

We need to show that if α and β are tautologies (from a shared language of course), then $\alpha \vDash \beta$ and $\beta \vDash \alpha$.

But given β is a tautology it is true on all valuations of its atoms, and hence – by a special case of the reasoning for (a*) – it is true on all valuations of the atoms of the relevant language. Hence there will be no valuation of the atoms of the relevant language which makes α true and makes β false, whatever α is. Hence $\alpha \vDash \beta$.

Similarly, given α is a tautology, $\beta \vDash \alpha$ whatever β is.

(2) *If* $\alpha \vDash \gamma$*, then* $\alpha, \beta \vDash \gamma$*.*

If $\alpha \vDash \gamma$, then no valuation of all the atoms in the relevant language makes α true and γ false, whence there is no valuation of all the atoms in the relevant language which makes α and also some β from the same language true while making γ false. Hence $\alpha, \beta \vDash \gamma$.

(3) *If* $\alpha, \beta \vDash \gamma$ *and* β *is a tautology, then* $\alpha \vDash \gamma$*.*

Observation: if β is a tautology, the valuations of the atoms of the relevant language which make α true are exactly the same valuations as make α and β both true.

Suppose then $\alpha, \beta \vDash \gamma$. Every valuation of the atoms of the language which makes α, β both true makes γ true. But if β a tautology, that's equivalent (by our observation) to saying that every valuation of the atoms of the language which makes α true makes γ true. Hence $\alpha \vDash \gamma$.

(4) *If* $\alpha \vDash \beta$ *and* $\beta \vDash \gamma$*, then* $\alpha \vDash \gamma$*.*

Given $\alpha \vDash \beta$ and $\beta \vDash \gamma$, every valuation which makes α true makes β true and every valuation which makes β true makes γ true, hence every valuation which makes α true makes γ (where in each case the valuations are those of all the atoms in the relevant language). Whence $\alpha \vDash \gamma$.

(5) *Suppose $\beta \approx \beta'$ (i.e. $\beta \vDash \beta'$ and $\beta' \vDash \beta$). Then for any wffs α, γ both (i) $\alpha \vDash \beta$ if and only if $\alpha \vDash \beta'$, and (ii) $\beta \vDash \gamma$ if and only if $\beta' \vDash \gamma$.*

Just use (4) repeatedly. Suppose we are given then $\beta \approx \beta'$. Then if $\alpha \vDash \beta$, we then have both $\alpha \vDash \beta$ and $\beta \vDash \beta'$ and so (by 4) $\alpha \vDash \beta'$. Likewise on the same supposition, if $\alpha \vDash \beta'$, we then have both $\alpha \vDash \beta'$ and $\beta' \vDash \beta$ and so (by 4) $\alpha \vDash \beta$. Which establishes (i).

(ii) is proved similarly.

(6) *Replacing a subformula of a wff by an equivalent expression results in a new wff equivalent to the original one.*

The rough idea is this. Suppose γ has the subformula α; then γ's value on a given valuation of atoms of the language will depend in part on α's value on that valuation. Replace α with α' and we get a new wff γ' whose valuation on that same valuation of atoms of the language will depend in the same way on α''s value on that valuation. But if α and α' are equivalent, their values will be the same: hence, since γ''s value depends on that common value in the same way as γ does, γ''s value will the same on that valuation as γ. And that applies whatever valuation of the atoms we choose.

(c*) *Some new notation. Alongside the use of lower-case Greek letters for individual wffs, it is common to use upper-case Greek letters such as 'Γ' (*Gamma*) and 'Δ' (*Delta*) to stand in for some wffs – zero, one, or many.*

Further, we use 'Γ, α' for the wffs Γ together with α. We also use '$\Gamma \cup \Delta$' for the wffs Γ together with the wffs Δ.

Now prove these generalized versions of the some of the claims in (b):

(1) *If $\Gamma \vDash \gamma$, then $\Gamma, \alpha \vDash \gamma$.*

If $\Gamma \vDash \gamma$, then no valuation of all the atoms in the relevant language makes all the wffs Γ true and γ false, whence no valuation of all the atoms in the relevant language makes all the wffs Γ and some α from the same language true and γ false; whence $\Gamma, \alpha \vDash \gamma$.

(2) *If $\Gamma, \alpha \vDash \gamma$ and α is a tautology, then $\Gamma \vDash \gamma$.*

Observation: if α is a tautology, the valuations of the atoms of the relevant language which make some wffs Γ all true are exactly the same valuations as make Γ, α all true.

Suppose then $\Gamma, \alpha \vDash \gamma$. Every valuation of the atoms of the language which makes Γ, α all true makes γ true. But if α a tautology, that's equivalent (by our observation) to saying that every valuation of the atoms of the language which makes the wffs Γ true makes γ true. Hence $\Gamma \vDash \gamma$.

(3) *If $\Gamma \vDash \beta$ and $\Delta, \beta \vDash \gamma$, then $\Gamma \cup \Delta \vDash \gamma$.*

Given (i) $\Gamma \vDash \beta$, every valuation which makes all of Γ true makes β true. Given (ii) $\Delta, \beta \vDash \gamma$, every valuation which makes all of Δ plus β true makes γ true. So take a valuation which makes all of Γ and Δ true. By (i) this makes β and all of Δ true, so by (ii) makes γ true. For short, $\Gamma \cup \Delta \vDash \gamma$.

(4) *State and prove a general version of (b5) which allows for inferences with multiple premisses.*

The general version we want is this: Suppose $\beta \approx \beta'$. Then for any wffs Γ and wff γ, (i) $\Gamma \vDash \beta$ if and only if $\Gamma \vDash \beta'$, and (ii) $\Gamma, \beta \vDash \gamma$ if and only if $\Gamma, \beta' \vDash \gamma$.

Let's this time prove part of (ii). So suppose $\Gamma, \beta \vDash \gamma$ and $\beta \approx \beta'$. Then we have $\beta' \vDash \beta$, and $\Gamma, \beta \vDash \gamma$. And then applying (3) $\Gamma, \beta' \vDash \gamma$. [Here β is the sole wff in the Γ place in (3), β' takes the place of β, Γ now takes the place of Δ – so $\Gamma \cup \Delta$ in (3) becomes the β plus the wffs Γ.]

The other part of (ii) is similar, and (i) is easy.

We will normally be interested in cases where we are dealing with only finitely many wffs Γ. *But would (1) to (3) still be true if* Γ *and/or* Δ *were infinitely many?*

Yes. Nothing depends on how many wffs Γ and/or Δ we are dealing with.

(d*) *Given the results*

(1′) $(\alpha \land \beta) \approx (\beta \land \alpha)$
(2′) $(\alpha \land (\beta \land \gamma)) \approx ((\alpha \land \beta) \land \gamma)$
(3′) $(\alpha \land (\beta \lor \gamma)) \approx ((\alpha \land \beta) \lor (\alpha \land \gamma))$

it can be said 'conjunction is commutative', 'conjunction is associative', 'conjunction distributes over disjunction'. Investigate and explain. What parallel claims apply to disjunction?

Wikipedia is surprisingly good if you need to look up definitions of simple logical or mathematical notions. Here's what it says about the property of being commutative:

> In mathematics, a binary operation is commutative if *changing the order of the operands does not change the result.* It is a fundamental property of many binary operations, and many mathematical proofs depend on it. Most familiar as the name of the property that says $3+4 = 4+3$ or $2 \times 5 = 5 \times 2$, the property can also be used in more advanced settings.

So a binary operation $\star$ is commutative if $x \star y$ and $y \star x$ are equal for any appropriate values of x and y. $\land$ is commutative as a truth-functional operation. So evidently is $\lor$, so $(\alpha \lor \beta) \approx (\beta \lor \alpha)$.

Here next is what Wikipedia says about the property of being associative, with a bit of editing:

> In mathematics, the associative property is a property of some binary operations.
>
> Within an expression containing two or more occurrences in a row of the same associative operator, the order in which the operations are performed does not matter as long as the sequence of the operands is not changed. That is, rearranging the parentheses in such an expression will not change its value. Consider the following equations:
>
> $$(2+3)+4 = 2+(3+4)$$
>
> $$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$
>
> Even though the parentheses were rearranged on each line, the values of the expressions were not altered. Since this holds true when performing addition and multiplication on any real numbers, it can be said that "addition and multiplication of real numbers are associative operations".

So a binary operation $\star$ is associative if $x \star (y \star z)$ and $(x \star y) \star z$ are equal for any appropriate values of x, y, z. $\land$ is associative as a truth-functional operation. So evidently is $\lor$, so $(\alpha \lor (\beta \lor \gamma)) \approx ((\alpha \lor \beta) \lor \gamma)$.

And here I'll rephrase something from Wikipedia about the property of being distributive.

> Suppose $\star$ and $\circ$ are two binary operations (defined over the same objects). Then
>
> 1. $\star$ is left-distributive over $\circ$ if $x \star (y \circ z) = (x \star y) \circ (x \star z)$,
> 2. $\star$ is right-distributive over $\circ$ if $(y \circ z) \star x = (y \star x) \circ (z \star x)$,
> 3. $\star$ is distributive over $\circ$ if it is both left and right distributive over $\circ$.
>
> Notice that when $\star$ is commutative, the three conditions above are logically equivalent.

So distributivity is a more complex property. For example, division is right-distributive over addition, but not left-distributive ($(y+z) \div x = (y \div x) + (z \div x)$, but in general $x \div (y+z) \neq (x \div y) + (x \div z)$).

Multiplication is distributive over addition ($x \times (y+z) = (x \times y) + (x \times z)$), but addition is not distributive over multiplication (in general, $x + (y \times z) \neq (x+y) \times (x+z)$). However, with conjunction and disjunction, the situation is symmetric. Not only is ($3'$) true, i.e. conjunction distributes over disjunction as truth-functions, but also disjunction distributes over conjunction. Thus, $(\alpha \lor (\beta \land \gamma)) \approx ((\alpha \lor \beta) \land (\alpha \lor \gamma))$.

(e*) *Find out more about the idea of 'working backwards' (touched on in Ch. 16) by looking at the online supplement on propositional truth trees.* See here.