Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 18: The truth-functional conditional

(a) *Suppose we are working in a* PL *language where '*P*' means* Putnam is a philosopher*, '*Q*' means* Quine is a philosopher*, etc. Translate the following as best you can:*

(1) *If either Quine or Putnam is a philosopher, so is Russell.*

$((\mathsf{Q} \lor \mathsf{P}) \to \mathsf{R})$

(2) *Only if Putnam is a philosopher is Russell one too.*

$(\mathsf{R} \to \mathsf{P})$

(3) *Quine and Russell are both philosophers only if Sellars is.*

$((\mathsf{Q} \land \mathsf{R}) \to \mathsf{S})$

(4) *Russell's being a philosopher is a necessary condition for Quine's being one.*

$(\mathsf{Q} \to \mathsf{R})$

To say (i) *A is a necessary condition for B* is to say that you can't have *B* without having *A*, so certainly implies (ii) *if not-A, then we can't have not-B*, or equivalently (ii′) *if B is to be true, A has to true too* and hence implies the corresponding formal wff (iii) $(\mathsf{B} \to \mathsf{A})$ is true.

But does (i) say *more* than (iii)? Arguably so – after all $(\mathsf{B} \to \mathsf{A})$ is true just so long as A is true, and yet it would seem *very* odd to say that when *A* happens to be true, it is a necessary condition for any old unrelated *B*! What's gone missing, it seems, in going from (i) to (iii) is the implication of *some* sort of genuine link between *A* and *B*. But in the PL language we have, with just a material conditional available, the given translation for (4), or its contrapositive, is the best we can do.

(5) *Russell's being a philosopher is a sufficient condition for Quine's being one.*

$(\mathsf{R} \to \mathsf{Q})$

To say (i) *A is a sufficient condition for B* is to say that having *A* true is enough for ensure that *B* holds as well, so certainly implies (ii) *if A is to be true, B has to true too* and hence implies the corresponding formal wff (iii) $(\mathsf{A} \to \mathsf{B})$ is true. But does (i) say *more* than (iii)? Surely so – after all $(\mathsf{A} \to \mathsf{B})$ is true just so long as B is true, and yet it would seem *very* odd to say that when *B* happens to be true, any old unrelated *A* is sufficient for *B*! Again, what's gone missing, it seems, in going from (i) to (iii) is the implication of *some* sort of genuine link between *A* and *B*. But in the language we have, with just a material conditional available, the given translation for (5) is also the best we can do.

(6) *Putnam is a philosopher if and only if Quine isn't.*

$((\neg\mathsf{Q} \to \mathsf{P}) \land (\mathsf{P} \to \neg\mathsf{Q}))$

The first clause translate the 'if' part, the second clause translates the 'only if' part.

(7) *Provided that Quine is a philosopher, Russell is one too.*

$(\mathsf{Q} \to \mathsf{R})$

It seems that 'provided that *A*, *B*' is typically equivalent to 'if *A* then *B*'.

(8) *Quine is not a philosopher unless Russell is one.*

$(\mathsf{Q} \to \mathsf{R})$

(8) is surely equivalent to *Quine is a philosopher only if Russell is one*, and hence the suggested formal rendition.

Now, if *not-A unless B* goes to the corresponding formal $(A \to B)$, plain *A unless B* should go to $(\neg A \to B)$, which is truth-functionally equivalent to $(A \lor B)$. That squares with our response to Exercises 10 (a) 9.

(9) *Only if either Putnam or Russell is a philosopher are both Quine and Sellars philosophers.*

$((\mathsf{Q} \land \mathsf{S}) \to (\mathsf{P} \lor \mathsf{R}))$

(b) *Assuming that we are dealing with a suitable* PL *language. Which of the following arguments* ought *to come out valid, assuming that '$\to$' is a reasonably good surrogate for 'if ..., then ...'? Which is tautologically valid?*

(1) $\mathsf{P},\ (\mathsf{P} \to \mathsf{Q}),\ (\mathsf{Q} \to \mathsf{R})\ \therefore\ \mathsf{R}$

Ought to be valid! Given P and *if P then Q* we can infer Q by modus ponens. Given *if Q then R* as well, we can infer R by another modus ponens, getting the conclusion as claimed.

A truth table confirms (1) is tautologically valid.

(2) $\neg\mathsf{R},\ (\mathsf{P} \to \mathsf{R}),\ (\mathsf{Q} \to \mathsf{P})\ \therefore\ \neg\mathsf{Q}$

Ought to be valid! Taking the premisses in the opposite order, given *if Q then P* and *if P then R* we of course can infer *if Q then R* (that's the transitivity of the conditional). And from that and the first premiss $\neg\mathsf{R}$, our conclusion follows by modus tollens.

A truth table confirms (2) is tautologically valid.

(3) $(\mathsf{P} \to \neg(\mathsf{Q} \lor \mathsf{R})),\ (\mathsf{Q} \to \mathsf{R}),\ (\neg\mathsf{R} \to \mathsf{P})\ \therefore\ (\mathsf{P} \to \mathsf{R})$

This doesn't look like a plausible argument! And it isn't tautologically valid – the line of the truth-table where $\mathsf{P} := \mathsf{T}$, $\mathsf{Q} := \mathsf{F}$, $\mathsf{R} := \mathsf{F}$ makes the premisses true and conclusion false.

You'd discover that by a brute truth-table test. But you could also reason as follows. Suppose there is a bad line, making the premisses true and conclusion false (as we suspect there will be!). Then, to make the conclusion false, we will need $\mathsf{P} := \mathsf{T}$, and $\mathsf{R} := \mathsf{F}$. To make the second premiss true, we will then also need $\mathsf{Q} := \mathsf{F}$. So there's nothing else to do but check that this assignment of values to the atoms makes the first and third premisses true too – which it does!

(4) $(\mathsf{P} \lor \mathsf{Q}),\ (\mathsf{P} \to \mathsf{R}), \neg(\mathsf{Q} \land \neg\mathsf{R})\ \therefore\ \mathsf{R}$

Ought to be valid! The second and third premisses reflect the claims that *if P then R* and that we can't have Q without also having R. So if we are also given that *P or Q*, we know that we can conclude R either way.

A truth table confirms (4) is tautologically valid.

(5) $(\mathsf{R} \to (\neg\mathsf{P} \lor \mathsf{Q})),\ (\mathsf{P} \land \neg\mathsf{R}) \therefore \neg(\neg\mathsf{R} \lor \mathsf{Q})$

This doesn't look like a plausible argument! And it isn't tautologically valid – a line of the truth-table where $\mathsf{P} := \mathsf{T}$, $\mathsf{R} := \mathsf{F}$ (and Q is either value) makes the premisses true and conclusion false.

You'd discover that by a brute truth-table test. But you could also reason as follows. Suppose there is a bad line, making the premisses true and conclusion false (as we suspect there will be!). Then, to make the second premiss true, we will need $\mathsf{P} := \mathsf{T}$, and $\mathsf{R} := \mathsf{F}$. But $\mathsf{R} := \mathsf{F}$ is enough to make the first premiss true and the conclusion false!

(6) $(\neg\mathsf{P} \lor \mathsf{Q}),\ \neg(\mathsf{Q} \land \neg\mathsf{R})\ \therefore\ (\mathsf{P} \to \mathsf{R})$

Ought to be valid! Suppose we are told that (i) *either not-P or Q* and (ii) *not(Q while not-R)*. Well, then, if P, by (i) we have Q, and from that and (ii) we can rule out *not-R*, i.e. conclude R.

A truth table confirms (6) is tautologically valid.

(7) $(\mathsf{P} \wedge \neg\mathsf{R}),\ (\mathsf{Q} \to \mathsf{R})\ \therefore\ \neg(\mathsf{P} \to \mathsf{Q})$

Ought to be valid! Suppose we are told that (i) *P and not-R*, (ii) *if Q then R*, and (iii) *if P then Q*. Then (i) gives us P, so from (iii) we get Q and then from (ii) we get R, contradicting (i). So (i) and (ii) together must rule out (iii), i.e. show *not-(if P then Q)*.

A truth table confirms (7) is tautologically valid.

(8) $\neg(\neg\mathsf{S} \to (\neg\mathsf{Q} \wedge \mathsf{R})),\ (\mathsf{P} \vee \neg\neg\mathsf{Q}),\ (\mathsf{R} \vee (\mathsf{S} \to \mathsf{P}))\ \therefore\ (\mathsf{P} \to \mathsf{S})$

This doesn't look like a plausible argument! And it isn't tautologically valid, as you'd discover by a brute truth-table test.

However, you could also reason as follows. Suppose there is a bad line, making the premisses true and conclusion false (as we suspect there will be!). Then, to make the conclusion false, we will need $\mathsf{P} \coloneqq \mathsf{T}$, and $\mathsf{S} \coloneqq \mathsf{F}$.

But $\mathsf{P} \coloneqq \mathsf{T}$ makes the second premiss true, while $\mathsf{S} \coloneqq \mathsf{F}$ makes $(\mathsf{S} \to \mathsf{P})$ true and hence makes the third premiss true. So can now just choose values of Q and R to make $(\neg\mathsf{Q} \wedge \mathsf{R})$ false, and hence make the first premiss true.

(c) *Which of the following are true for all α, β, γ in a PL language and why? Which of the true claims correspond to true claims about the vernacular (bi)conditional?*

(1) *If $\alpha, \beta \vDash \gamma$ then $\alpha \vDash (\beta \to \gamma)$.*

This is true. Fix on some wffs α, β, γ. If every relevant valuation which makes α and β true makes γ true, then every valuation which makes α true either doesn't make β true or makes γ true, i.e. every valuation which makes α true makes $(\beta \to \gamma)$ true.

And it corresponds to an intuitive truth: given that α and β logically entail γ, then α entails that *if* you have β *then* it will be true that γ.

(2) $((\alpha \wedge \beta) \to \gamma) \mathrel{=}\!\!\mid\!\vDash (\alpha \to (\beta \to \gamma))$.

This is true. Take some wffs α, β, γ. You can use a truth-table to confirm that (whatever values those wffs take) $((\alpha \wedge \beta) \to \gamma)$ always takes the same value as $(\alpha \to (\beta \to \gamma))$.

The left-to-right direction corresponds to an intuitive truth. If γ is true given that α and β are, then if α is indeed true, then if β is true as well, then γ will be true. Similarly for the right-to-left direction.

(3) $((\alpha \vee \beta) \to \gamma) \mathrel{=}\!\!\mid\!\vDash ((\alpha \to \gamma) \vee (\beta \to \gamma))$.

It is easy to check that $((\alpha \vee \beta) \to \gamma) \vDash ((\alpha \to \gamma) \vee (\beta \to \gamma))$. Indeed we have the stronger claim $((\alpha \vee \beta) \to \gamma) \vDash ((\alpha \to \gamma) \wedge (\beta \to \gamma))$ (why should we expect that?).

But the converse doesn't hold, i.e. $((\alpha \to \gamma) \vee (\beta \to \gamma)) \vDash ((\alpha \vee \beta) \to \gamma)$ is false. Just replace α and γ with false wffs and β with a true wff, and the left-hand wff will be true, and the right-hand wff false.

(4) *If $\vDash (\alpha \to \beta)$ and $\vDash (\beta \to \gamma)$, then $\vDash (\alpha \to \gamma)$.*

Obviously true. Take some wffs α, β, γ. If every valuation makes both $(\alpha \to \beta)$ and $(\beta \to \gamma)$ true it will make $(\alpha \to \gamma)$ true.

But the converse doesn't hold, i.e. $((\alpha \to \gamma) \vee (\beta \to \gamma)) \vDash ((\alpha \vee \beta) \to \gamma)$ is false. Just replace α and γ with false wffs and β with a true wff, and the left-hand wff will be true, and the right-hand wff false.

(5) *If $\vDash (\alpha \to \beta)$ and $\vDash (\alpha \to \neg\beta)$, then $\vDash \neg\alpha$.*

Obviously true. Take some wffs α, β. If $\vDash (\alpha \to \beta)$, then every valuation which makes α true makes β true. If $\vDash (\alpha \to \neg\beta)$, then every valuation which makes α true makes β false. So no valuation can make α true (since no valuation can make β and $\neg\beta$ true). Hence $\vDash \neg\alpha$.

And this is again what we should intuitively expect: if it a logical truth both that if α then β and if α than $\neg\beta$, then it will be a logical truth that α is false.

(6) $\vDash (\alpha \leftrightarrow \alpha)$
(7) $(\alpha \leftrightarrow \beta) \vDash (\beta \leftrightarrow \alpha)$.
(8) $(\alpha \leftrightarrow \beta), (\beta \leftrightarrow \gamma) \vDash (\alpha \leftrightarrow \gamma)$.

Three easy results corresponding to three intuitive claims about the logic of the biconditional.

(9) *If* $\vDash \alpha \leftrightarrow \beta$ *then* α *and* β *are tautologically consistent.*

False! Suppose α and β are both contradictions; then $\vDash \alpha \leftrightarrow \beta$ but they are not tautologically consistent (no valuation makes them true together).

(10) *If* $\vDash \alpha \leftrightarrow \neg\beta$ *then* α *and* β *are tautologically inconsistent.*

True! If $\vDash \alpha \leftrightarrow \neg\beta$ then every valuation which makes α true makes β false – hence there is no valuation which makes α and β true together, i.e. they are tautologically inconsistent.

(d*) *On alternative languages for propositional logic:*

(1) *Suppose the language* PL_1 *has just the connectives* $\rightarrow$ *and* $\neg$ *(with the same interpretation as before). Show that disjunction and conjunction can be expressed in* PL_1*. Conclude that* PL_1 *has an expressively adequate set of built-in connectives.*

This question is starred not because it is difficult or involved, but simply to highlight that here are two (easy!) facts that you ought to know.

For (1), just note that in PL, $(\alpha \vee \beta)$ is equivalent to $(\neg\alpha \rightarrow \beta)$ and $(\alpha \wedge \beta)$ is equivalent to $\neg(\alpha \rightarrow \neg\beta)$. Now we knew already that any truth-function can be expressed in PL using a wff using conjunction, disjunction and negation. So now we know that truth-function can be expressed in PL using a wff using just the conditional and negation (by replacing each conjunction or disjunction with an equivalent using the conditional and negation). So in fact the limited resources of PL_1 will be enough to express every truth-function.

(2) *Consider too the variant language* PL_2 *whose only logical constants are* $\rightarrow$ *and the absurdity constant* $\bot$*. Show that in* PL_2 *we can introduce a negation connective so that* $\neg\alpha$ *is shorthand for* $(\alpha \rightarrow \bot)$*. Conclude that* PL_2 *is also expressively adequate.*

Check that claim that $\neg\alpha$ is equivalent to $(\alpha \rightarrow \bot)$. It then follows from the result in (1) that every truth-function can be expressed by some wff using just the conditional and negation, that (2) every truth-function can be expressed by some wff using just the conditional and the absurdity constant.