7 Quantifier complexity

Wffs of the language L_A come in different degrees of *quantifier complexity*. We can distinguish, for a start, so-called Δ_0 , Σ_1, and Π_1 wffs. Later, in §11.2, we will note that the standard Gödel sentence that sort-of-says 'I am unprovable' is a Π_1 wff. This is important – it means that, while the Gödel sentence might be very long and messy, there is also a good sense in which it is logically really quite simple. Why? What is a Π_1 wff? This short chapter explains.

7.1 Q knows about bounded quantifications

We often want to say that all/some numbers less than or equal to some bound have a particular property. We can express such claims in formal arithmetics like Q and PA by using wffs of the shape $\forall \mathsf{x}(\mathsf{x} \leq \tau \to \varphi(\mathsf{x}))$ and $\exists \mathsf{x}(\mathsf{x} \leq \tau \ \wedge \ \varphi(\mathsf{x}))$, where $\mathsf{x} \leq \tau$ is just short for $\exists \mathsf{v}(\mathsf{v} + \mathsf{x} = \tau)$ (see §5.7), and τ can stand in for any term (not just some numeral) so long as it doesn't contain v free. It is standard to further abbreviate such wffs by $(\forall \mathsf{x} \leq \tau)\varphi(\mathsf{x})$ and $(\exists \mathsf{x} \leq \tau)\varphi(\mathsf{x})$ respectively.

Now note that we have easy results like these:

1. For any n, $\mathsf{Q} \vdash \forall \mathsf{x}(\{\mathsf{x} = \overline{0} \vee \mathsf{x} = \overline{1} \vee \ldots \vee \mathsf{x} = \overline{\mathsf{n}}\} \leftrightarrow \mathsf{x} \leq \overline{\mathsf{n}})$.

2. For any n, if $\mathsf{Q} \vdash \varphi(\overline{0}) \wedge \varphi(\overline{1}) \wedge \ldots \wedge \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.

3. For any n, if $\mathsf{Q} \vdash \varphi(\overline{0}) \vee \varphi(\overline{1}) \vee \ldots \vee \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.

Such results show that Q – and hence a stronger theory like PA – 'knows' that bounded universal quantifications (with fixed number bounds) behave like finite conjunctions, and that bounded existential quantifications (with fixed number bounds) behave like finite disjunctions.

7.2 Δ_0 wffs

Let's say that

Defn. 26. *An L_A wff is Δ_0 iff it can be built up from the non-logical vocabulary of L_A plus $\leq$ (defined as before), using the familiar propositional connectives, the identity sign, but only* bounded *quantifications.*

So, a Δ_0 wff is just like a quantifier-free L_A wff, except that we are now allowed the existential quantifiers used in defining occurrences of $\leq$, and we can allow ourselves to wrap up some finite conjunctions into bounded universal quantifications, and similarly wrap up some finite disjunctions into bounded existential quantifications.

It should be no surprise to hear this:

Theorem 18. *We can effectively decide the truth-value of any Δ_0 sentence.*

We won't give a full-dress proof. But, roughly speaking, we can unpack bounded quantifications into conjunctions or disjunctions (perhaps in a number of stages, if bounded quantifiers are nested one inside another). And then we are left with an equivalent wff built up using propositional connectives from basic expressions of the form $\sigma = \tau$ and $\sigma \leq \tau$ (for quantifier-free σ and τ) – and we can compute the truth values of such basic expressions.

Since we can mechanically decide whether $\varphi(\overline{\mathsf{n}})$ when that is Δ_0, this means that we can mechanically determine whether a Δ_0 open wff $\varphi(\mathsf{x})$ is satisfied by a given number n. In other words, a Δ_0 open wff $\varphi(\mathsf{x})$ will express a decidable property of numbers. Likewise a Δ_0 open wff $\varphi(\mathsf{x}, \mathsf{y})$ will express a decidable numerical relation.

Now, since (i) Theorem 13 tells us that even Q can correctly decide all quantifier-free L_A sentences, (ii) Theorem 16 tells us that Q also knows about the relation $\leq$, and (iii) Q knows that quantifications with numeral bounds behave just like conjunctions/disjunctions, the next result won't be a surprise either:

Theorem 19. Q *(and hence* PA*) can correctly decide all Δ_0 sentences.*

Again, we won't spell out the argument here (enthusiasts can see the proof of Theorem 11.2 in *IGT2*).

7.3 Σ_1 and Π_1 wffs

We next say that

Defn. 27. *An L_A wff is Σ_1 if it is (or is logically equivalent to) a Δ_0 wff preceded by zero, one, or more unbounded* existential *quantifiers. And a wff is Π_1 if it is (or is logically equivalent to) a Δ_0 wff preceded by zero, one, or more unbounded* universal *quantifiers.*

As a mnemonic, it is worth remarking that 'Σ' in the standard label 'Σ_1' comes from an old alternative symbol for the existential quantifier, as in $\Sigma x Fx$ – that's a Greek 'S' for '(logical) sum'. Likewise the 'Π' in 'Π_1' comes from corresponding symbol for the universal quantifier, as in $\Pi x Fx$ – that's a Greek 'P' for '(logical) product'. And the subscript '1' in 'Σ_1' and 'Π_1' indicates that we are dealing with wffs which start with *one* block of similar quantifiers, respectively existential quantifiers and universal quantifiers.[1]

So a Σ_1 wff says that some number (pair of numbers, etc.) satisfies the decidable condition expressed by its Δ_0 core; likewise a Π_1 wff says that every number (pair of numbers, etc.) satisfies the decidable condition expressed by its Δ_0 core.

To check understanding, pause to make sure you understand why

1. The negation of a Δ_0 wff is still Δ_0.
2. A Δ_0 wff is also Σ_1 and Π_1.
3. The negation of a Σ_1 wff is Π_1.
4. The negation of a Π_1 wff is Σ_1.

(Recall the rules for exchanging the order of quantifiers and negations!) And let's note the following easy result:

Theorem 20. Q *(and hence* PA*) can prove any true* Σ_1 *sentences (is '*Σ_1*-complete').*

Proof. Take, for example, a sentence of the type $\exists x \exists y \varphi(x, y)$, where $\varphi(x, y)$ is Δ_0. If this sentence is true, then for some pair of numbers m, n, the Δ_0 sentence $\varphi(\overline{m}, \overline{n})$ must be true. But then by Theorem 19, Q proves $\varphi(\overline{m}, \overline{n})$ and hence $\exists x \exists y \varphi(x, y)$, by existential introduction.

Evidently the argument generalizes for any number of initial quantifiers, which shows that Q proves all truths which are (or are provably-in-Q equivalent to) some Δ_0 wff preceded by one or more unbounded existential quantifiers. □

7.4 A remarkable corollary

Our last theorem looks entirely straightforward and unexciting, but it has an immediate corollary which is much more interesting:

Theorem 21. *If* T *is a consistent theory which includes* Q*, then every* Π_1 *sentence that it proves is true.*

Proof. Suppose T proves a *false* Π_1 sentence φ. Then $\neg\varphi$ will be a *true* Σ_1 sentence. But in that case, since T includes Q and so is 'Σ_1-complete', T will also prove $\neg\varphi$, making T inconsistent. Contraposing, if T is consistent, any Π_1 sentence it proves is true. □

[1] Just for the record, we can keep on going, to consider wffs with greater and greater quantifier complexity. So, we say a Π_2 wff is (or is logically equivalent to) one that starts with *two* blocks of quantifiers, a block of universal quantifiers followed by a block of existential quantifiers followed by a bounded kernel. Likewise, a Σ_2 wff is (equivalent to) one that starts with two blocks of quantifiers, a block of existential quantifiers followed by a block of universal quantifiers followed by a bounded kernel. And so it goes, up the so-called *arithmetical hierarchy* of increasing quantifier complexity. But we won't need to consider higher up that the first levels of the arithmetical hierarchy here.

Which is, in its way, a quite remarkable observation. It means that we don't have to fully *believe* a theory T – i.e. don't have to accept that all its theorems are *true* on the interpretation built into T's language – in order to use it to establish that some Π_1 arithmetic generalization is true.

For example, with some minor trickery, we can state Fermat's Last Theorem as a Π_1 sentence. And famously, Andrew Wiles has shown how to prove Fermat's Last Theorem using some *extremely* heavy-duty infinitary mathematics. Now we see, intriguingly, that we actually don't have to believe that this infinitary mathematics is *true* – whatever exactly that means when things get so very wildly infinitary! – but only that it is *consistent*, to take Wiles as establishing that the Π_1 arithmetical claim which is the Theorem is true. Remarkable!

7.5 Intermediate arithmetics

We said at the beginning of the previous chapter that, in moving on from the very weak arithmetics BA and Q to consider first-order PA, we were jumping over a whole family of theories of intermediate strength. We can now briefly describe those intermediate theories: they are the ones we get by restricting the quantifier complexity of suitable instances of the induction schema.

For example, $\mathsf{I\Sigma_1}$ is the theory we get by taking the first six axioms of PA (§6.5) plus every instance of the Induction Schema

$$[\varphi(0) \land \forall x(\varphi(x) \to \varphi(Sx))] \to \forall x\varphi(x),$$

where $\varphi(x)$ is now an open Σ_1 wff of L_A that has 'x' free.

There is *technical* interest in knowing how much a theory like $\mathsf{I\Sigma_1}$ can prove (as we will see in §16.1). But do such theories have any *conceptual* interest? After all, we gave reasons in §6.3 for being generous with induction: we asked, if $\varphi(x)$ expresses a genuine arithmetical property, how can induction fail for $\varphi(x)$?

But which L_A open wffs $\varphi(x)$ (with one free variable) *do* express genuine properties? Previously we took it that they all do (even if, in the general case, we may not be able to decide whether a given number n has the property or not): that is why we said that any such wff $\varphi(x)$ is 'suitable' for appearing in an instance of the Induction Schema (see §6.3 again). But backtrack a moment from that cheerful assumption: suppose you are a *very* stern constructivist who thinks that an expression $\varphi(x)$ only *really* makes sense if it is Δ_0 and so we can effectively decide whether or not it holds true of a given number (or if it is Σ_1 and we can prove it true of a given number when it is). *Then* you can reasonably want to restrict the induction principle to suitable instances using only Δ_0 (or Σ_1) expressions. But it would take us far too far afield to explore the merits of such positions here!