

Logic: A Study Guide

Peter Smith

Logic Matters

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Preface

Before I retired from the University of Cambridge, it was my greatest good fortune to have secure, decently paid, university posts for forty years in leisurely times, with almost total freedom to follow my interests wherever they meandered. Like most of my contemporaries, for much of that time I didn't really appreciate how *very* lucky I was. This Study Guide to logic textbooks is my attempt to give a little back by way of heartfelt thanks.

The Guide is aimed at two main groups of readers – philosophers who want to go on beyond their first introductory logic course to learn some more serious logic, and mathematicians wanting to get to grips with an under-taught but exciting area of mathematics. So there two separate introductory chapters written for the two kinds of readers. But why not two separate Guides? Because it would be difficult to decide quite what should go where. After all, a number of philosophers develop interests in more mathematical corners of the broad field of logic. And a number of mathematicians find themselves becoming interested in more foundational/conceptual issues. Rather than impose artificial divisions, I provide here a single but wide-ranging menu for everyone to choose from as their interests dictate. So . . .

Don't be scared off by the Guide's length! This is due both to its breadth of coverage and also to its starting just half a step beyond 'baby logic' and then going quite a long way down the road towards state-of-the-art treatments. Different readers will therefore want to jump on and off the bus at different stops. Simply choose the sections which are most relevant to your background and your interests, and you will be able to cut the Guide down to very much more manageable proportions. There's a lot of signposting to enable you to pick your way through.

However, if you are hoping for help with *very* elementary logic (perhaps as encountered by philosophers in their first formal logic courses), then let me say straight away that this Guide is not designed for you. The only chapters that even briefly mention logic at this level are the two Introductions. All the rest is about rather more advanced – and eventually *very* much more advanced – material.

Most of the recommendations in this Guide point to published *books*. True, there are quite a lot of on-line lecture-notes that university teachers have made available. Some of these are excellent. However, they do tend to be terse, and

often *very* terse (as entirely befits material originally intended to support a lecture course). They are usually not as helpful as fully-worked-out book-length treatments for students needing to teach themselves.

So where can you find the books mentioned here? Well, they should in fact be held by any large-enough university library which has been trying over the years to maintain core collections in mathematics and philosophy (and if the local library is too small, books should be borrowable through some inter-library loans system).

Since it is assumed that you will by default be using library copies of books, I have *not* made cost or being currently in print a significant consideration.

However, I have marked with one star* books that are available new or second-hand at a reasonable price (or at least are unusually good value for the length and/or importance of the book).

I have marked with two stars** those books for which e-copies are freely and legally available, and links are provided.

Where articles or encyclopaedia entries have been recommended, these can almost always be freely downloaded, again with links supplied.

We must pass over in silence the question of using a certain particularly well-known and well-stocked copyright-infringing PDF repository. That is between you and your conscience. I could not possibly comment . . .

This re-titled Guide expands and revises the mid 2020 edition of my much-downloaded *Teach Yourself Logic*. Many thanks, then, to all those who have commented on different versions of *TYL* over more than a decade: and as always, further comments and suggestions are always gratefully received.

1 Introduction for philosophers

1.1 Why this Guide for philosophers?

It is an odd phenomenon, and a very depressing one too. Logic, beyond a very elementary introductory level, is taught less and less, at least in UK philosophy departments. Fewer and fewer philosophers with serious logical interests seem to get appointed to teaching posts.

Yet logic itself is, of course, no less exciting and rewarding a subject than it ever was, and the amount of good formally-informed work in philosophy is ever greater as time goes on. Moreover, logic is far too important to be left entirely to the mercies of technicians from maths or computer science departments with different agendas (who often reveal an insouciant casualness about basic conceptual issues that will matter to the more philosophical student).

So how is a real competence in logic to be passed on if there are not enough courses, or indeed if there are none at all? It seems that many beginning graduate students in philosophy will need to teach themselves logic from books, either solo or by organizing their own study groups (local or online).

In a way, this is perhaps no real hardship; there are some wonderful books written by great expositors out there. But *what* to read and work through? Logic books can have a *very* long shelf life, and you shouldn't at all dismiss older texts when starting out on some topic area: so there's more than a sixty year span of publications to select from. There are literally hundreds of formal logic books that might feature somewhere in a Study Guide such as this.

Philosophy students evidently need a Guide if they are to find their way around the very large literature old and new, with the aim of teaching themselves enjoyably and effectively. This is my attempt to provide one.

1.2 Assumed background

So what do you need to bring to the party, if you are going to tackle some of the books recommended in the main body of this Guide? You should be well prepared if you have worked through a book like one of the following:

1. Peter Smith, *Introduction to Formal Logic*** (2nd edition, CUP, 2020; corrected version now freely downloadable from logicmatters.net/ifl). The first edition was the first year text in Cambridge for a decade.

1 Introduction for philosophers

Written as an accessible teach-yourself book, it covers basic propositional and predicate logic. The first edition did logic ‘by trees’. This new second edition instead focuses on natural deduction; but material on trees is still available as online supplements.

2. P. D. Magnus, Tim Button, and others, *forallx*** (now part of the Open Logic Project and freely available from forallx.openlogicproject.org.) Also focuses on natural deduction, and is considerably brisker and less expansive than my book.
3. Nicholas Smith *Logic: The Laws of Truth* (Princeton UP 2012). This excellent introduction is very clearly written, and has many virtues (particularly if you like your texts to go rather slowly and discursively). The first two parts of the book focus on logic by trees. But the third part ranges wider, including a brisk foray into natural deduction. There are some extras too, going significantly beyond ‘baby logic’. And there is a rich mine of end-notes. The book’s website is at njjsmith.com/philosophy/lawsoftruth/.

However, if your initial logic course was based on some *very* elementary text book like Sam Guttenplan’s *The Languages of Logic*, Howard Kahane’s *Logic and Philosophy*, or Patrick Hurley’s *Concise Introduction to Logic*, then you might struggle with the initial suggestions in the main part of the present Guide. But this will inevitably vary a lot from person to person. So just make a start and see how things work out; if the going gets too tough, backtrack to read one of the books like my own, skipping quickly over what you already know.

(And if you have only done an ‘informal logic’ or ‘critical reasoning course’, then you’ll very probably need to read a good introductory formal logic text before tackling the more advanced topics covered in the body of this Guide.)

1.3 But do you really need more logic?

Before plunging in, however, it is probably worth pausing to ask yourself whether you, as a budding philosopher, really *do* want to pursue your logical studies much further, assuming you have done a decent introductory course on formal logic. You certainly will want to know more logic, if you aim to be a philosopher of mathematics for example. But for many purposes, you might well survive by reading the relevant parts of this very useful book:

Eric Steinhart, *More Precisely: The Math You Need to Do Philosophy** (Broadview, 2nd edition 2017, with a [companion website](#)). The author writes: “The topics presented . . . include: basic set theory; relations and functions; machines; probability; formal semantics; utilitarianism; and infinity. The chapters on sets, relations, and functions provide you with all you need to know to apply set theory in any branch of philosophy. The chapter of machines includes finite state machines, networks of

machines, the game of life, and Turing machines. The chapter on formal semantics includes both extensional semantics, Kripkean possible worlds semantics, and Lewisian counterpart theory. The chapter on probability covers basic probability, conditional probability, Bayes theorem, and various applications of Bayes theorem in philosophy. . . . The chapters on infinity cover recursive definitions, limits, countable infinity, Cantor’s diagonal and power set arguments, uncountable infinities, the aleph and beth numbers, and definitions by transfinite recursion. *More Precisely* is designed both as a text book and reference book to meet the needs of upper level undergraduates and graduate students. It is also useful as a reference book for any philosopher working today.”

Steinhart’s book is admirable, and will give many philosophers a handle on some technical ideas going well beyond first-year logic and which they really should know just a little about, without all the hard work of doing a full mathematical logic course. What’s not to like? It could be enough for you. And then, if there indeed turns out to be some particular area (modal logic, for example) that seems especially germane to your particular philosophical interests, you can always consult the relevant section of this Guide to find out more.

Still, there will always be discriminating philosophers who will want to study logic in much greater depth than a book like Steinhart’s can provide!

1.4 How to prove it

Experience shows that being able to handle e.g. natural deduction proofs in a formal system doesn’t always translate into being able to construct good *informal* proofs. For example, one of the few meta-theoretic results that might be met in a first logic course is the expressive completeness of the set of formal connectives $\{\wedge, \vee, \neg\}$. The proof of this result is based on a very simple proof-idea. But some philosophy students who ace the part of the end-of-course exam asking for *formal* proofs inside a deductive system find themselves all at sea when asked to outline a correct *informal* proof elaborating that proof idea.

Another example: it is only too familiar to find philosophers introduced to set notation not being able even to make a start on giving a correct informal proof that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$ if and only if $a = a'$ and $b = b'$.

Well, if you *are* one of those who jumped through the formal hoops but were unclear about how to set out elementary mathematical proofs from the ‘meta-theory’ of baby logic or from very introductory set theory, then this is exactly what you need:

Daniel J. Velleman, *How to Prove It: A Structured Approach** (CUP, 3rd edition, 2019). From the Preface: “Students . . . often have trouble the first time that they’re asked to work seriously with mathematical proofs, because they don’t know ‘the rules of the game’. What is expected of

1 Introduction for philosophers

you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs.”

There are chapters on the propositional connectives and quantifiers, and on key *informal* proof-strategies for using them; there are chapters on relations and functions, a chapter on mathematical induction, and a final chapter on infinite sets (countable vs. uncountable sets). This is a truly excellent student text, and working through it from the beginning could be exactly what you need to get you prepared for the serious study of logic. Even if you were one of those comfortable with the informal proofs, you will probably still profit from skipping and skimming through the book (perhaps paying especial attention to the chapter on mathematical induction).

2 Introduction for mathematicians

2.1 Why this Guide for mathematicians?

Mathematics students soon pick up a passing acquaintance with some *very* basic notions about sets and some logical symbolism. But usually they get no systematic understanding of formal logic. They may be told early on that set theory provides a foundation for mathematics (in some sense), but again most students never encounter even the outlines of a full-blown set theory. Indeed, there are full university maths courses in good UK universities with precisely *zero* courses offered on the core mathematical logic curriculum – first-order logic and basic model theory, the theory of computability, set theory. And the situation can be equally patchy elsewhere.

So if you want to teach yourself some logic, where should you start? What are the topics you might want to cover? What textbooks are likely to prove accessible, engaging, and rewarding to work through? As I remarked in the Introduction for philosophers, logic books can have a very long shelf life, and you shouldn't at all dismiss older texts when starting out. So there's more than a sixty year span of publications to select from, and that's hundreds of books. This Guide – or at least, the sections on the core curriculum – will give you pointers to some of the best for self-study.

True, this is written by someone who, apart from a few guest mini-courses, has taught in philosophy departments and who has never been a research mathematician. Which no doubt gives a distinctive tone to the Guide, and also explains why it occasionally ranges into some areas of logic of likely to be most of interest to philosophers. Still, mathematics remains my first love, and most of the books I recommend on core topics are very definitely paradigm *mathematics* texts. So I won't be leading you astray.

2.2 Assumed background

There is no specific mathematical background you need before tackling the entry-level books surveyed in Part I of this Guide. They don't presuppose very much 'mathematical maturity', so you really can just dive in.

However, I did recommend to philosophers a quite excellent book (written by a mathematician for mathematicians) which talks in a very introductory way

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about logical notation and its use in proofs – namely *Daniel J. Velleman, *How to Prove It: A Structured Approach* (CUP, 3rd edition, 2019). This could serve as a very useful introduction to informal logical ideas.

Suppose, however, that you are already know how to use the so-called propositional connectives symbolized ‘ \wedge ’ or ‘ $\&$ ’, ‘ \vee ’, ‘ \neg ’ or ‘ \sim ’, and ‘ \rightarrow ’. Suppose too that you are comfortably familiar with the use of formal quantifiers, e.g. as in $\forall\epsilon\exists\delta$ used in stating the ϵ/δ definition of continuity. Then you probably won’t strictly speaking *need* Velleman’s introduction. Though it certainly could do no harm to pause now to speed-read the book, slowing down when you need to.

3 Using this Guide

This third preliminary chapter says something about how to use this Guide, and outlines the Guide's overall structure.

3.1 Strategies for reading logic books

(a) We cover a great deal of ground in this Guide – from entry level books to decidedly advanced texts – which is one reason for its initially daunting length. But there is another reason, connected to a point which I now want to highlight:

I very strongly recommend tackling an area of logic by reading a series of books which *overlap* in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can't stress this advice too much (which, in my experience, applies equally to getting to grips with any new area of mathematics). This approach will really help to reinforce and deepen understanding as you encounter and re-encounter the same material at different levels, coming at it from somewhat different angles, with different emphases.

Exaggerating only a little, there are many instructors who say 'This is the textbook we are using/here is my set of notes: take it or leave it'. But you will *always* gain from looking at some overlapping texts. (When responding to student queries on a question-and-answer internet site, I'm repeatedly struck by how much puzzlement would be resolved by taking the occasional quick look outside the course textbook/lecturer's notes.) The multiple overlaps in coverage in the reading lists in later chapters, which help make the Guide as long as it is, are therefore fully intended. They also mean that you should always be able to find options that suit your degree of mathematical competence.

To repeat: you will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Yes, do very carefully read one or two central texts, choosing books at a level that works for you. But do also cultivate the crucial further habit of judiciously skipping and skimming through a number of other works so that you can build up a good overall picture of an area seen from various angles of approach.

3 Using this Guide

(b) While we are talking about strategies for self-teaching, I should add a quick remark on the question of doing exercises.

Mathematics is, as they say, not merely a spectator sport: so you should try some of the exercises in the books as you read along, in order to check and reinforce comprehension. On the other hand, don't obsess about this, and do concentrate on the exercises that look interesting and/or might deepen understanding. And note too that some authors have the irritating(?) habit of burying quite important results among the exercises, mixed in with routine homework. It is therefore always a good policy to skim through the exercises in a book even if you don't plan to work on answers to very many of them.

Oddly – even in these days where books can have websites to support them – it isn't that common for logic texts to have very detailed worked solutions to exercises available. I will try to highlight those authors who are more helpful than usual in this respect.

3.2 How this Guide is structured

'Logic', in the broad sense, is a big field. Different constituencies will be particularly concerned with different areas. Core 'classical first-order logic' is basic by anyone's lights. But after that, interests can diverge.

For example, modal logic is of considerable interest to some philosophers, not so much to mathematicians, though parts of this sub-discipline are of great concern to some computer scientists. Set theory (which falls within the purview of mathematical logic, broadly understood) is an active area of research interest in mathematics, but – because of its (supposed) foundational status – even quite advanced results can be of interest to philosophers too. Type theory started off as a device of philosophy-minded logicians looking to avoid the paradoxes: it seems to have become primarily the playground of computer scientists. The incompleteness theorems are relatively elementary results of the theory of computable functions, but are of particular conceptual interest to philosophers. And so it goes.

In this Guide, I'm going to have to let the computer scientists largely look after themselves. Our main focus is going to be on the mathematical logic curriculum of most concern to both philosophers and mathematicians, together with some extras of particular interest to philosophers – if only because that's what I know a little about. Here, then, is an overview map of the territory we do cover in the three parts of this Guide.

Part I: Core Topics in Mathematical Logic starts with chapters on classical first-order logic and a little introductory model theory. Next, we look at formal arithmetic, the theory of computable functions, and proofs of epochal results such as Gödel's incompleteness theorems. Finally in this part, there is an introductory chapter on set theory – informal set theory, basic notions of cardinals and ordinals, constructions of number systems

in set theory, the role of the axiom of choice, etc., and the formal axiomatization of ZFC. These chapters taken together survey the sort of ground covered by classic texts like Elliott Mendelson's *Introduction to Mathematical Logic*.

Part II: Other Logics starts by looking at some variant logics that everyone should encounter early in their logical studies, namely second-order logic (already touched on in Part I) and intuitionistic logic. We then look at other logics of particular philosophical interest, varieties of modal logics, relevant logic, plural logic, free logic.

Part III: Going Further in Mathematical Logic picks up a central topic not traditionally covered in the core curriculum, namely proof theory. And then it goes on to offer suggestions for more advanced reading on the topics of Part I, further exploring model theory, computability and formal arithmetics, and set theory. There also pointers to some other topics. This part of the Guide is for specialist graduate students (among philosophers) and for final year undergraduate/beginning graduate students (among mathematicians).

The Guide divides up the broad field of logic in a pretty conventional way: but of course, even these 'horizontal' divisions into different subfields can in places be a little arbitrary. And the 'vertical' division between the entry-level coverage in Part I and the further explorations of the same areas in Part III is necessarily going to be a lot more arbitrary. Everyone will agree (at least in retrospect!) that e.g. the elementary theory of ordinals and cardinals belongs to the basics of set theory, while explorations of 'large cardinals' or independence proofs via forcing are decidedly advanced. But in most areas, there are fewer natural demarcation lines between the entry-level basics and more advanced work. Still, it is surely very much better to have *some* such structuring than to heap everything together.

Each chapter starts with a Overview, which briskly introduces its topic(s). I have then put the main recommendations within a chapter into what strikes me as a sensible reading order of increasing difficulty (those with stronger mathematical backgrounds might sometimes want to try starting in the middle of a list). Some additional books are listed in asides or postscripts.

One comment. Earlier versions of the precursor to this Guide used also to have a substantial final part considering some of 'The Big Books on mathematical logic' (meaning typically broader-focus books that cover first-order logic together with one or more subfields from the further menu of mathematical logic). These books vary a lot in level and coverage, and topic-by-topic are rarely to be preferred to more focused books. However, they can provide very useful consolidating/amplifying reading. This old supplement to the main body of the Guide is now available online as a separate [Appendix](#) which I hope to continue adding to. Alternatively, for individual webpages on those texts and a number of additional reviews, visit the Logic Matters [Book Notes](#).

3.3 Choices, choices

So what has guided my choices of what books to recommend?

Different people find different expository styles congenial. What is agreeably discursive for one reader might be irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good on conceptual details and good at explaining the motivation for the technicalities while avoiding needless complications, excessive hacking through routine detail, or misplaced ‘rigour’. Given the choice, I tend to prefer a treatment that doesn’t rush too fast to become too general, too abstract, and thereby obscures intuitive motivation: this is surely what we want in books to be used for self-study. (There’s a certain tradition of masochism in older maths writing, of going for brusque formal abstraction from the outset with little by way of explanatory chat: this is quite unnecessary in other areas, and just because logic is all about formal theories, that doesn’t make it any more necessary here.)

The selection of books in the following chapters no doubt reflects these tastes. But overall, I don’t think that I have been very idiosyncratic: indeed, in many respects I have probably been rather conservative in my choices. So nearly all the books I recommend will very widely be agreed to have significant virtues (even if other logicians would have different preference-orderings).

Finally, the earliest versions of this Guide kept largely to positive recommendations: I didn’t pause to explain the reasons for the then absence of some well-known books. This was partly due to considerations of length which have now quite gone by the wayside; but also I wanted to keep the tone enthusiastic, rather than to start criticizing or carping.

However, enough people kept asking what I think about some alternative X , or asking why the old warhorse Y wasn’t mentioned, for me to change my mind. So I have just occasionally added some reasons why I *don’t* particularly recommended certain books.

Part I

Core Topics in Mathematical Logic

4 First order logic

So, let's get down to business! This chapter starts with a quick overview of the topics we will be treating as belonging to the basics of first-order logic (or predicate logic, quantificational logic, call it what you will: I'll use 'FOL' for short). Then there are some main recommendations for texts covering these topics, followed by some suggestions for parallel and further reading. The chapter ends with some additional comments, mostly responding to frequently asked questions.

A note to philosophers. If you have carefully read and mastered a substantial introductory logic text for philosophers such as Nick Smith's, or even my own, you will already be familiar with (basic versions of) a significant amount of the material covered in this chapter. However, the big change is that you will now begin to see the perhaps familiar material being re-presented in the sort of mathematical style and with the sort of rigorous detail that you will necessarily encounter more and more as you progress in logic. You do need to start feeling entirely comfortable with this mode of presentation at an early stage. So do work through the topics again, now with more mathematical precision.

4.1 FOL overview: basic topics

FOL deals with deductive reasoning that turns on the use of 'propositional connectives' like *and*, *or*, *if*, *not*, and on the use of 'quantifiers' like *every*, *some*, *no*. But in ordinary language (and even in informal mathematics) these logical operators work in quite complex ways, introducing the kind of obscurities and possible ambiguities we want to avoid in logically transparent arguments. What to do? From the time of Aristotle, logicians have used a 'divide and conquer' strategy that involves introducing restricted, tidied-up, 'formalized' languages. We tackle a stretch of reasoning by first regimenting it into a suitable unambiguous formalized language with tidier logical operators, and then we can evaluate the reasoning once recast into this more transparent form. This way, we have a division of labour. There's the task of working out the intended structure of the original argument as we render it into a clear formal language. Then there's the separate business of assessing the validity of the resulting nicely regimented argument.

In FOL, therefore, we use appropriate formal languages which contain, in

particular, (i) tidily disciplined surrogates for the connectives *and*, *or*, *if*, *not* (standardly symbolized \wedge , \vee , \rightarrow and \neg) and (ii) replacements for the ordinary language quantifiers (roughly, using $\forall x$ for *every x is such that ...*, and $\exists y$ for *some y is such that ...*). Although the fun really starts once we have the quantifiers in play, it is useful to develop FOL in two corresponding main stages. So:

- (i) Typically, we start by introducing propositional languages whose built-in logical apparatus comprises just the propositional connectives, and discuss the resulting propositional logic. This gives us a very manageable setting in which to first encounter a whole range of logical concepts and strategies.
- (ii) We next develop full first-order languages which also have the apparatus of quantification – such languages still have very sparse logical apparatus but are in fact rich enough to regiment almost all mathematical reasoning. We can then explore the logic of arguments rendered into such languages.

We can usefully say just a bit more about the stages of this project now – though don't worry, of course, if you don't yet fully grasp the import of every point:

(i.a) We begin by looking at the *syntax* of propositional languages, defining what count as well-formed formulas of such languages. If you have already encountered such languages, you should now get to know how to prove various things about them that might seem obvious and that you perhaps previously took for granted – for example, that ‘bracketing works’ to avoid ambiguities, meaning that every well-formed formula has a unique parsing.

(i.b) On the *semantic* side, you need to understand the idea of a *valuation* for a propositional language: we start with an assignment of truth-values, *true* vs *false*, to the atomic components of our languages, and then explain how to evaluate complex sentences involving the connectives by using the ‘truth-functional’ interpretation of the connectives.

With these basic semantic ideas in play, we can now define the relation of semantic entailment, where the sentences Γ semantically entail the sentence φ when no possible valuation will make all the sentences Γ true without making φ true too. We then get to explore some of the key properties of this entailment relation, and learn how to calculate when the relation holds.

(i.c) Different textbook presentations of (i.a) and (i.b) will be pretty similar; but now the paths fork. For the next topic will be a deductive *proof-system* for propositional logic, and there is a variety of such systems to choose from. For a start – forgive the arm-waving! – we have

1. Old-school ‘axiomatic’ systems. (Proofs are simple linear sequences of formulas set out e.g. as a single vertical column; each formula in a proof after the initial premisses is either an instance of a logical axiom or follows from what's gone before by one of a couple of basic rules of inference.)

4 First order logic

2. Natural deduction done Fitch-style. (We are this time allowed lots of rules of inference and so don't need logical axioms; in particular, we are now allowed to use forms of inference where we make temporary suppositions for the sake of argument – as when we temporarily assume φ , show it entails a contradiction and conclude that *not*- φ . To keep track of which temporary suppositions are in play at a given point of an argument, we use a multi-column layout, with the line of formulas in the proof jumping to and fro. The premisses sit at the top of the home column, and the desired conclusion at the bottom, but we *indent* the argument into a new column each time a new temporary supposition is made, and cancel the indentation when that supposition is dropped again.)
3. Natural deduction done Gentzen-style. (Again, as in informal reasoning, we are allowed lots of rules of inference and don't need logical axioms. But this time, we set out the proof in the form of a root-at-the-bottom-tree. Premisses appear at the top of branches; proof-branches can join, as when we combine φ on one branch and ψ on another to infer their conjunction $\varphi \wedge \psi$; and the final conclusion is at the root of the tree. Again we are allowed to use inferences which discharge temporary suppositions, but we will need a different device to keep track of them.)
4. 'Truth trees' or 'semantic tableaux'. (Instead of arguing directly from given premisses to our desired conclusion, we 'work backwards'. So we start by assuming that the premisses are true and the conclusion is false. Put these assumptions at the top of an upside-down tree. We aim to show that these initial assumptions are contradictory (so indeed, if the given premisses are true, so is the conclusion). We need to set our working in a downwards-branching tree because we will need to explore possible alternatives at choice points.)
5. Sequent calculi. (Also due to Gentzen, our working is again set out as a root-at-the-bottom-tree. But this time the 'nodes' of the tree are decorated not with single formulas but with so-called sequents – in one form, these are expressions like $\Gamma : \Delta$ which tell us that if the formulas Γ are all true then at least one of the formulas Δ is true.)

These different types of proof-system (or rather their specific versions for 'classical' propositional logic) are equivalent – meaning that, given some premisses, we can warrant the same conclusions in each system. But they differ considerably in their intuitive appeal and user-friendliness, as well as in some of their more technical features.

In fact, Fitch proofs and truth trees are easiest for beginners to manipulate in order to produce working formal proofs – which is why elementary logic books for philosophers usually introduce one or other (or both) systems. But, as we will see, mathematical logic text books at the next level of sophistication usually focus on the first and third types of system, on axiomatic systems and Gentzen-style natural deduction. True, axiomatic systems in their raw state can initially

be pretty horrible to use – but proofs can be expedited once you learn some basic dodges (like the use of the so-called ‘Deduction Theorem’). Gentzen-style proofs are by comparison very elegant, though they still take some getting used to compared to Fitch-style proofs which stick rather closer to informal patterns of reasoning. Finally, the sequent calculus really comes into its own in more advanced work on so-called proof theory.

At some point, the educated logician will want to find about *all* these proof styles – at the very least, you should get a general sense of how they respectively work, and come to appreciate the interrelations between them.

(i.d) We now have two different ways of defining what’s a deductively good argument in propositional logic. We said that some premisses Γ semantically entail a conclusion φ if every possible valuation which makes Γ true makes φ true. We can now say that some premisses Γ entail the conclusion φ in the proof-system S if there is an S -type derivation of the conclusion φ from the premisses Γ (or a derivation of the sequent $\Gamma : \{\varphi\}$). Of course we want these two approaches to fit together. We want our favoured proof-system S to be *sound* – it shouldn’t give false positives. In other words, if there is an S -derivation of φ from the premisses Γ , then φ really *is* semantically entailed by the premisses Γ . We also want our favoured proof-system S to be *complete* – we want it to capture all the correct semantic entailment claims. In other words, if φ is semantically entailed by the premisses Γ , then there is indeed some S -derivation of φ from the premisses Γ .

So we will want proofs of soundness and completeness for our favoured proof-system S . These results will hold no terrors! However, in establishing soundness and completeness for propositional logics we encounter useful strategies which can later be beefed-up to give us soundness and completeness results for stronger logics.

(ii.a) Now for FOL (predicate logic, quantificational logic). Again, *syntax* first. And a story needs to be told in particular about why the syntax of the ‘quantifier/variable’ expressions of generality in formal logic is significantly different from the syntax of ‘every’ or ‘some’ in ordinary language. There are also decisions to be made about allowing the use of so-called ‘free variables’ and/or having ‘dummy names’ or ‘parameters’ in the language.

(ii.b) Turning to *semantics*: the first key idea we need is that of a *structure* (a domain of objects equipped with some relations and/or functions). And, crucially, you need to grasp the idea of an interpretation of a language in such a structure; you’ll need to understand how such an interpretation generates a unique valuation (a unique assignment of truth-values) for every sentence of the interpreted language. A proper formal semantic story with the bells and whistles needed to cope with quantifiers is non-trivial.

With these semantic ideas to hand, you can again define a relation of semantic entailment, now for FOL expressions. The FOL sentences Γ semantically entail φ when no interpretation in any appropriate structure can make all the sentences Γ true without making φ true too.

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You'll again need to know some of the basic properties of this entailment relation.

(ii.c) Next, we will want to explore a proof system for FOL. Corresponding to the five basic types of system we mentioned for propositional logic, you can again encounter five different types of proof system, with their varying attractions. And it should be said that even when we have chosen the basic type we want to work with, there remain other important choices to make – though we should end up each time with an equivalent system which warrants the same arguments. Again, you'll want at some point to find out about all these different styles of proof: but (as we said) we will start by looking at axiomatic and (one flavour of) natural deduction systems.

(ii.d) As with propositional logic, we will want *soundness* and *completeness* proofs which show that our chosen proof systems for FOL don't overshoot (giving us false positives) or undershoot (so we can't derive some semantically valid entailments). And at *this* point, it might be said, the study of FOL becomes really interesting: the completeness proof in particular involves more sophisticated ideas than anything we have met before. It also has intriguing corollaries, as we will see when we move from our initial investigation of FOL to begin so-called 'model theory', about which much more in the next chapter.

4.2 The main recommendations on FOL

Unsurprisingly, there is a *very* long list of texts which introduce FOL. But the point of this Guide is to choose! So here are my top recommendations, starting with one-and-a-third stand-out books which, taken together, make an excellent introduction:

1. Ian Chiswell and Wilfrid Hodges, *Mathematical Logic* (OUP 2007). This nicely written text is very approachable. It is written by mathematicians primarily for mathematicians. However, it is only one notch up in actual difficulty from some introductory texts for philosophers like mine or Nick Smith's, though – as its title might suggest – it does have a notably more mathematical 'look and feel'. It should in fact be entirely manageable for self study by philosophers and mathematicians alike (philosophers can skip over a few more mathematical illustrations).

The briefest headline news is that authors explore a Gentzen-style natural deduction system. But by building things up in three stages – so after propositional logic, they consider an interesting fragment of first-order logic before turning to the full-strength version – they make e.g. proofs of the completeness theorem for first-order logic unusually comprehensible. For a more detailed description see my [book note](#) on C&H.

Very warmly recommended, then. For the moment, you only *need*

read up to and including §7.7 (under two hundred pages). But having got that far, you might as well read the final couple of sections and the Postlude too! The book has brisk solutions to some of the exercises. (A demerit mark, though, to OUP for not publishing C&H more cheaply.)

Next, you should complement C&H by reading the first third of

2. Christopher Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic*** (1st edn by Leary alone, Prentice Hall 2000; inexpensive 2nd edn Milne Library 2015; and now [freely available for download](#)).

There is a great deal to like about this book. Chs. 1–3, in either edition, do indeed make a friendly and helpful introduction to FOL, this time done in axiomatic style. At this stage you could stop reading after §3.2, which means you will be reading just 86 pages.

Unusually, L&K dive straight into a full treatment of first-order logic without spending an introductory chapter or two on propositional logic. But that happily means (in the present context) that you won't feel that you are labouring through the very beginnings of logic yet one more time than is really necessary – this book therefore dovetails very nicely with C&H.

Again written by mathematicians, some illustrations of ideas can presuppose a smattering of background mathematical knowledge; but philosophers will miss very little if they occasionally have to skip an example (and the curious can always resort to Wikipedia, which is quite reliable in this area, for explanations of some mathematical terms). The book ends with extensive answers to exercises.

I like the overall tone of C&H very much indeed, and say more about this admirable book in another [book note](#).

For additional motivation for the quantifier/variable notation, see Chapters 26–28 of Peter Smith, *Introduction to Formal Logic*** (2nd edn. downloadable from www.logicmatters.net/ifl). For more explanation of the Gentzen-style proof system treated perhaps just a little quickly by C&H, see Chapter 9, 'Natural Deduction' of Richard Zach et al., *Sets, Logic, Computation*** (downloadable from the Open Logic Project, <https://slc.openlogicproject.org>).

Next, here's an alternative to the C&H/L&K pairing which is very approachable and can also be warmly recommended:

3. Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) is explicitly designed for self-study. Read Chs. 1 to 5 (you could skip §§4.4 and 4.5, leaving them until you turn to elementary model theory).

While C&H and the first third of L&K together cover overlapping material twice, Goldrei – in a comparable number of pages – covers very

similar ground once. So this is a somewhat more gently-paced book, allowing Goldrei to be more expansive about fundamentals, and to give a lot of examples and exercises with worked answers to test comprehension along the way. A very great deal of thought has gone into making this text as helpful as possible. Some may find it occasionally goes a bit too slowly and carefully, though I'd say that this is erring on the right side in an introductory book: if you struggle slightly with the alternative reading, or just want a comfortingly manageable text, you should find this exceptionally accessible. Or you might just warm more to Goldrei's style anyway.

Like L&K, Goldrei uses an axiomatic system (which is one reason why, on balance, I still recommend starting with C&H instead). As with C&H and L&K, I like the tone and approach a great deal.

Supplement Goldrei with a brief glance at a natural deduction proof system, e.g. again as in the Open Logic Project's *Sets, Logic, Computation* Ch. 9.

These three main recommended books, by the way, have all had very positive reports over the years from student users.

4.3 Some parallel/additional reading

The material covered in the last section is so very fundamental, and the alternative options so very many, that I really do need to say at least something about a few other books. So in this section I will list – in very rough order of difficulty/sophistication – a small handful of further texts which could well make for useful parallel or additional reading. Then in the following section, I will mention some other books I've been asked about.

I'll begin with a book written by a philosopher for philosophers:

4. David Bostock, *Intermediate Logic* (OUP 1997). From the preface: "The book is confined to . . . what is called first-order predicate logic, but it aims to treat this subject in very much more detail than a standard introductory text. In particular, whereas an introductory text will pursue just one style of semantics, just one method of proof, and so on, this book aims to create a wider and a deeper understanding by showing how several alternative approaches are possible, and by introducing comparisons between them." So Bostock does indeed usefully introduce you to tableaux ('truth trees') and an Hilbert-style axiomatic proof system and natural deduction and even a sequent calculus as well (as noted before, it is important eventually to understand what is going on in these different kinds of proof-system). So Bostock ranges more widely but less deeply than e.g. Goldrei. Anyone could profit from at least a quick browse of his Part II to pick up the headline news about the various approaches.

Bostock eventually touches on issues of philosophical interest such as free logic which are not often dealt with in other books at this level.

Still, the discussions mostly remain at much the same level of conceptual/mathematical difficulty as e.g. my own introductory book. He proves completeness for tableaux in particular, which I always think makes the needed construction seem particularly natural. *Intermediate Logic* should therefore be, as intended, particularly accessible to philosophers who haven't done much formal logic before and should, if read in parallel, help ease the transition to coping with the more mathematical style of the books recommended in the last section.

To repeat, unlike our main recommendations, Bostock does discuss tableaux ('truth trees'). If you are a philosopher, you may well have already encountered them in your introductory logic course. If not, here's a short and painless introduction to trees for propositional logic which will also give you the basic idea: Peter Smith, [Propositional Truth Trees](#).

Next, even though it is giving a second bite to an author we've already met, I must mention a rather different discussion of FOL:

5. Wilfrid Hodges, 'Elementary Predicate Logic', in the *Handbook of Philosophical Logic*, Vol. 1, ed. by D. Gabbay and F. Guenther, (Kluwer 2nd edition 2001). This is a slightly expanded version of the essay in the first edition of the *Handbook* (read that earlier version if this one isn't available), and is written with Hodges's usual enviable clarity and verve. As befits an essay aimed at philosophically minded logicians, it is full of conceptual insights, historical asides, comparisons of different ways of doing things, etc., so it very nicely complements the more conventional textbook presentations of C&H, L&K and/or Goldrei.

Read at this stage the very illuminating first twenty short sections.

Now, as a follow up to C&H, I recommended L&K's *A Friendly Introduction* which uses an axiomatic system. As an alternative, here is an older and in its day much used text which should certainly be very widely available:

6. Herbert Enderton, *A Mathematical Introduction to Logic* (Academic Press 1972, 2002). This also focuses on a Hilbert-style axiomatic system, and is often regarded as a classic of exposition. However, it does strike me as somewhat more difficult than L&K, so I'm not surprised that students often report finding it a bit challenging *if used by itself as a first text*. Still, it is an admirable and very reliable piece of work which you should be able to cope with well if used as a supplementary second text, e.g. after you have tackled C&H.

Read up to and including §2.5 or §2.6 at this stage. Later, you can finish the rest of that chapter to take you a bit further into model theory. For more about this classic, see [this book note](#).

I'll also mention – though this time with a little hesitation – another much used text which has gone through multiple editions and should also be in any library; it is a useful natural-deduction based alternative to C&H. Later chapters

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of this book are also mentioned later in this Guide as possible reading for more advanced work, so it could be worth making early acquaintance with . . .

7. Dirk van Dalen, *Logic and Structure** (Springer, 1980; 5th edition 2012). The chapters up to and including §3.2 provide an introduction to FOL via natural-deduction. The treatment *is* often approachable and written with a relatively light touch. However – and this explains my hesitation – it has to be said that the book isn't without its quirks and flaws and inconsistencies of presentation (though perhaps you have to be an alert and rather picky reader to notice and be bothered by them). Still, the coverage and general approach is good.

Mathematicians should be able to cope readily. I suspect, however, that the book would occasionally be tougher going for philosophers if taken from a standing start – which is another reason why I have recommended beginning with C&H instead. (See my [more extended review](#) of the whole book.)

Going up a level of mathematical sophistication, we get to absolute classic, short but packed with good things:

8. Raymond Smullyan, *First-Order Logic** (Springer 1968, Dover Publications 1995). This is terse, but those with a taste for mathematical elegance can certainly try Parts I and II, just a hundred pages, after the initial recommended reading in the previous section. This beautiful little book is the source and inspiration of many modern treatments of logic based on tree/tableau systems. Not always easy, especially as the book progresses, but a delight for the mathematically minded.

Finally, taking things in a new direction, don't be put off by the title of

9. Melvin Fitting, *First-Order Logic and Automated Theorem Proving* (Springer, 1990, 2nd ed. 1996). This is a wonderfully lucid book by a renowned expositor. Yes, at various places in the book there are illustrations of how to implement various algorithms in Prolog. But either you can easily pick up the very small amount of background knowledge about Prolog that's needed to follow everything that is going on (and that's quite a fun thing to do anyway) or you can just skip those implementation episodes.

As anyone who has tried to work inside an axiomatic system knows, proof-discovery for such systems is often hard. Which axiom schema should we instantiate with which wffs at any given stage of a proof? Natural deduction systems are nicer. But since we can, in effect, make any new temporary assumption we like at any stage in a proof, again we still need to keep our wits about us if we are to avoid going off on useless diversions. By contrast, tableau proofs (a.k.a. tree proofs) can pretty much write themselves even for quite complex FOL arguments, which is why I used to introduce formal proofs to students that way (in teaching

tableaux, we can largely separate the business of getting across the idea of formality from the task of teaching heuristics of proof-discovery). And because tableau proofs very often write themselves, they are also good for automated theorem proving. Fitting explores both the tableau method and the related so-called resolution method – yes, a sixth style of proof! – in this exceptionally clearly written book.

This book’s emphasis is, then, rather different from most of the other recommended books. So I initially hesitated to mention it here in this Guide. However, I do think that the fresh light thrown on first-order logic makes the detour through this book *vaut le voyage*, as the Michelin guides say. (If you don’t want to take the full tour, however, there’s a nice introduction to proofs by resolution in Shawn Hedman, *A First Course in Logic* (OUP 2004): §1.8, §§3.4–3.5.)

4.4 Other treatments?

I will end this chapter by responding to a variety of Frequently Asked Questions, mostly questions raised in response to earlier versions of the Guide. Occasionally, I have to be pretty negative.

Hold on! What about an introductory text with a more proof-theoretic slant? Well, it’s true that there are some questions about systems of FOL which can be tackled at a quite introductory level, yet which aren’t addressed by any of the readings so far mentioned. Suppose we are working in some formal system and (i) have so far proved α and also proved β ; then (ii) we can (rather boringly!) infer the conjunction $\alpha \wedge \beta$ (remember ‘ \wedge ’ means *and*). Now, suppose later in the proof (iii) we appeal to that conjunction $\alpha \wedge \beta$ and infer α . We wouldn’t have gone *wrong*; but obviously the detour has been *pointless*, given that at stage (i) we could already prove α . So there is evident interest in the question of how to eliminate such detours and other pointless digressions from proofs. Gentzen famously started the ball rolling in his discussions of how to ‘normalize’ proofs in his natural deduction systems, and he showed how normalization results can be used to derive other important properties of the proof systems.

For a first encounter with this sort of topic, you could look at Jan von Plato’s *Elements of Logical Reasoning** (CUP, 2014). This is based on the author’s lectures for an introductory course. But a lot of material is touched on in a relatively short compass as von Plato talks about a range of different natural deduction and sequent calculi; I suspect that, without any classroom work to round things out, this book might not be as easily accessible as the author intends. But suppose you have already met one system of natural deduction (e.g., as in C&H), and now want to know more about ‘proof-theoretic’ aspects of this and related systems. Suppose, for example, that you want to know about variant ways of setting up ND systems, about proof-search, about the relation with so-called sequent calculi, etc. Then this book makes a reasonably clear and approachable start. However, my own recommendation would be to concentrate

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for the moment on the core topics covered by the books mentioned so far, and then later dive into proof theory proper, covered in Chapter ??.

A blast from the past: What about Mendelson? Somewhat to my surprise, perhaps the most frequent question I used to get asked in response to early versions of the Guide is ‘But what about Mendelson, Chs. 1 and 2’? Well, Elliot Mendelson’s *Introduction to Mathematical Logic* (Chapman and Hall/CRC 6th edn 2015) was first published when I was a student and the world was a great deal younger. The book was I think the first modern textbook of its type (so immense credit to Mendelson for that), and I no doubt owe my whole career to it – it got me through trips!

It seems that some others who learnt using the book are in their turn still using it to teach from. But let’s not get sentimental! It has to be said that the book in its first incarnation was often brisk to the point of unfriendliness, and the basic look-and-feel of the book hasn’t changed a great deal as it has run through successive editions. Mendelson’s presentation of axiomatic systems of logic are quite tough going, and as the book progresses in later chapters through formal number theory and set theory, things if anything get somewhat less reader-friendly. Which certainly doesn’t mean the book won’t repay battling with. But unsurprisingly, fifty years on, there are many rather more accessible and more amiable alternatives for beginning serious logic. Mendelson’s book is a landmark worth visiting one day, but I can’t recommend starting there. For a little more about it, [see here](#).

(As an aside, if you *do* want an old-school introduction from roughly the same era, you might more enjoy Geoffrey Hunter, *Metalogic** (Macmillan 1971, University of California Press 1992). This is not groundbreaking in the way e.g. Smullyan’s *First-Order Logic* is, nor is it as comprehensive as Mendelson: but it was an exceptionally good textbook from a time when there were few to choose from, and I still regard it with admiration. Read Parts One to Three at this stage. And if you are finding it rewarding reading, then do eventually finish the book: it goes on to consider formal arithmetic and proves the undecidability of first-order logic, topics we revisit in §??. Unfortunately, the typography – from pre-L^AT_EX days – isn’t at all pretty to look at: this can make the book’s pages initially appear rather unappealing. But in fact the treatment of an axiomatic system of logic is extremely clear and accessible. It might be worth blowing the dust off your library’s copy!)

Two more standard mathematical logic texts: what about Ebbinghaus, Flum and Thomas? what about Rautenberg? We now turn to H.-D. Ebbinghaus, J. Flum and W. Thomas, *Mathematical Logic* (Springer, 2nd edn. 1994). This is the English translation of a book first published in German in 1978, and appears in a series ‘Undergraduate Texts in Mathematics’, which indicates the intended level. The book is often warmly praised and is (I believe) quite widely used. But revisiting it, I can’t find myself wanting to recommend it as a good place to start, either for philosophers or for mathematicians. The core material on the syntax and semantics of first-order logic in Chs 2 and 3 is presented more accessibly

and more elegantly elsewhere. And the treatment of a sequent calculus Ch. 4 strikes me as poor, with the authors mangling some issues of principle and failing totally to capture the elegance that a sequent calculus can have. For more on this book, see this [book note](#).

Wolfgang Rautenberg's *A Concise Introduction to Mathematical Logic* (Springer, 2nd edn. 2006) has some nice touches. But its first hundred pages on FOL are probably *too* concise to serve most readers as a first introduction; and its preferred formal system is not a 'best buy' either. §1.5 on applications of the compactness theorem for propositional logic is adding to your reading list, though.

The very latest thing: what about the Open Logic Text? This is an entirely admirable, collaborative, open-source, enterprise inaugurated by Richard Zach, and very much work in progress. You can download the latest full version from [this page](#).

In an earlier version of this Guide, I said that "although this is referred to as a textbook, it is perhaps better regarded as a set of souped-up lecture notes, written at various degrees of sophistication and with various degrees of more book-like elaboration." My sense is that the chapters on propositional and quantificational logic – mixed into *Sets, Logic, Computation*** – have been expanded considerably, and are now much more book-like. They still, however, have some rather idiosyncratic features (for example why *start* with the LK system of sequent calculus?). But my main worry is that the discussions probably still go a little too fast to work as a purely stand-alone introduction for initial self-study without the benefit of lecture support. But your mileage may vary. And certainly, these notes could be *very* useful for reinforcing/revision.

Designed for philosophers: Why not The Logic Book? What about Sider? What about Bell, DeVidi and Solomon? Many US philosophers have had to take courses based on *The Logic Book* by Merrie Bergmann, James Moor and Jack Nelson (first published by McGraw Hill in 1980; a sixth edition was published – at a quite ludicrous price – in 2013). I doubt that those students much enjoyed the experience! This is a large book, over 550 pages, starting at about the level of my introductory book, and going as far as metalogical results like a full completeness proof for FOL, so its coverage overlaps with the books mentioned in the last section. But while reliable enough, it all strikes me, like some other readers who have commented, as *very* dull and laboured, and often rather unnecessarily hard going. You can certainly do better.

Theodore Sider – a very well-known philosopher – has written a text called *Logic for Philosophy** (OUP, 2010) aimed at philosophers, which I've repeatedly been asked to comment on. The book in fact falls into two halves. The second half (about 130 pages) is on modal logic, and I will return to that in Chapter ???. The first half of the book (almost exactly the same length) is on propositional and first-order logic, together with some variant logics, so is very much on the topic of this chapter. But while the coverage of modal logic is quite good, I can't at all recommend the first half of this book: I explain why in [another book note](#). True, a potentially attractive additional feature of this part of Sider's book is

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that it does contain brief discussions about e.g. some non-classical propositional logics, and about descriptions and free logic. But remember all this is being done in just 130 pages, which means that things are whizzing by very fast, so the breadth of Sider's coverage here goes with far too much superficiality. If you want some breadth, Bostock's book is still very much to be preferred.

A potential alternative to Bostock at about the same level, and which can initially look promising, is John L. Bell, David DeVidi and Graham Solomon's *Logical Options: An Introduction to Classical and Alternative Logics* (Broadview Press 2001). This book covers a lot pretty snappily – for the moment, just Chapters 1 and 2 are relevant – and some years ago I used it as a text for second-year seminar for undergraduates who had used my own tree-based book for their first year course. But many students found the exposition too terse, and I found myself having to write very extensive seminar notes. If you want some breadth, you'd again do better sticking with more expansive Bostock.

And I have *still* only touched on a small proportion of books we could mention! In particular, the other Big Books on Mathematical Logic of course have treatments of the relevant material: see the Guide's [Appendix](#) for more detailed discussion of some options. However, I think none of these Big Books hit quite the right level and pace to make them ideal for a *first* encounter with basic FOL, at least for many readers.

But I don't want to finish on a negative note: so finally ...

Puzzles galore: What about some of Smullyan's other books? I have already warmly recommended Smullyan's 1968 classic *First-Order Logic*. He went on to write some classic and very accessible texts on Gödel's theorem and on recursive functions, which we'll be mentioning later. But as well as these, Smullyan wrote many 'puzzle' based-books aimed at a wider audience, including the justly famous 1981 *What is the Name of This Book?** (Dover Publications reprint, 2011).

More recently, he wrote *Logical Labyrinths* (A. K. Peters, 2009). From the blurb: "This book features a unique approach to the teaching of mathematical logic by putting it in the context of the puzzles and paradoxes of common language and rational thought. It serves as a bridge from the author's puzzle books to his technical writing in the fascinating field of mathematical logic. Using the logic of lying and truth-telling, the author introduces the readers to informal reasoning preparing them for the formal study of symbolic logic, from propositional logic to first-order logic, ... The book includes a journey through the amazing labyrinths of infinity, which have stirred the imagination of mankind as much, if not more, than any other subject."

Smullyan starts, then, with puzzles, e.g. of this kind: you are visiting an island where there are Knights (truth-tellers) and Knaves (persistent liars) and then in various scenarios you have to work out what's true from what the inhabitants say about each other and the world. And, without too many big leaps, he ends with first-order logic (using tableaux), completeness, compactness and more. This is no substitute for standard texts, but – for those with a taste for being led up

to the serious stuff via sequences of puzzles – an entertaining and illuminating supplement.

Smullyan's later *A Beginner's Guide to Mathematical Logic** (Dover Publications, 2014) is more conventional. The first 170 pages are relevant to FOL. A rather uneven read, it seems to me, but again perhaps an engaging supplement to the texts recommended above.

5 Elementary model theory

5.1 From first-order logic to elementary model theory

The completeness theorem is the first high point – the first mathematically serious result – in a course in first-order logic; and some elementary treatments more or less stop there. Many introductory texts, however, continue just a little further with some first steps into model theory. It is clear enough what needs to come next: discussions of the so-called compactness theorem (also called the ‘finiteness theorem’), of the downward and upward Löwenheim-Skolem theorems, and of their implications. There’s less consensus, however, about what other introductory model-theoretic topics you should tackle at an early stage.

As you will see when you dive into the reading, you will very quickly meet claims that involve infinite cardinalities, and there also occasional references to the axiom of choice. Now in fact, even if you haven’t yet done an official set theory course, you may well have picked up all you need to know at least in order to begin model theory. If you have met Cantor’s proof that infinite collections come in different sizes, and if you have been warned to take note when a proof involves making an infinite series of choices, you will probably know enough. And anyway, Goldrei’s chapter recommended in a moment in fact has a brisk section on the ‘Set theory background’ needed at this stage. (If that’s *too* brisk, then perhaps do a skim read of e.g. Paul Halmos’s very short *Naive Set Theory**, or one of the other books mentioned at the beginning of §?? below.)

What to read on model theory, then? The very first volume in the prestigious and immensely useful Oxford Logic Guides series is Jane Bridge’s very compact *Beginning Model Theory: The Completeness Theorem and Some Consequences* (Clarendon Press, 1977) which neatly takes the story onwards just a few steps from the reading on FOL mentioned in our §4.2 above. The coverage strikes me as exemplary for a short first introduction for logicians. But the writing, though very clear, is rather terse in an old-school way; and the book looks like photo-reproduced typescript, and which makes it rather unpleasant to read. What, then, are the alternatives?

Two of the introductions to FOL that I mentioned in §4.3 have treatments of some elementary model theory. Thus there are fragments of model theory in §2.6 of Herbert Enderton’s *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), followed by a discussion in §2.8 of non-standard analysis: but this, for

our purposes here, is perhaps too little done too fast. Dirk van Dalen's *Logic and Structure** (Springer, 1980; 5th edition 2012) covers rather more model-theoretic material in more detail in his Ch. 3. You could read the first section for revision on the completeness theorem, then tackle §3.2 on compactness, the Löwenheim-Skolem theorems and their implications, before moving on to the action-packed §3.3 which covers more model theory including non-standard analysis again, and indeed touches on slightly more advanced topics like 'quantifier elimination'. However, my top votes for help with the first steps in elementary model theory go elsewhere:

1. I have already sung the praises of Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument** (Springer, 2005) for the accessibility of its treatment of FOL in the first five chapters. You can now read his §§4.4 and 4.5 (which I previously said you could skip) and then Ch. 6 on 'Some uses of compactness' to get a very clear introduction to some model theoretic ideas.

In a little more detail, §4.4 introduces some axiom systems describing various mathematical structures (partial orderings, groups, rings, etc.): this section could be particularly useful to philosophers who haven't really met the notions before. Then §4.5 introduces the notions of substructures and structure-preserving mappings. After proving the compactness theorem in §6.1 (as a corollary of his completeness proof), Goldrei proceeds to use it in §§6.2 and 6.3 to show various theories can't be finitely axiomatized, or can't be nicely axiomatized at all. §6.4 introduces the Löwenheim-Skolem theorems and some consequences, and the following section introduces the notion of diagrams and puts it to work. The final section, §6.6 considers issues about categoricity, completeness and decidability. All this is done with the same admirable clarity as marked out Goldrei's earlier chapters.

If you then want to go just a bit further, what you probably need next is the rather more expansive

2. María Manzano, *Model Theory*, Oxford Logic Guides 37 (OUP, 1999).

This book aims to be an introduction at the kind of intermediate level we are currently concerned with. And standing back from the details, I do very much like the way that Manzano structures her book. The sequencing of chapters makes for a very natural path through her material, and the coverage seems very appropriate for a book at her intended level. After chapters about structures (and mappings between them) and about first-order languages, she proves the completeness theorem again, and then has a sequence of chapters on core model-theoretic notions and proofs. This is all done tolerably accessibly (just half a step up from Goldrei, perhaps).

True, the discussions at some points would have benefitted from rather

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more informal commentary, motivating various choices. And there are some infelicities. But overall, Manzano's text should work well and there is no evident competitor book at this level. See this [Book Note on Manzano](#) for more details.

And now the path forks. Philosophers will certainly want to tackle the following recently published book, which strikes me as a particularly impressive achievement:

3. Tim Button and Sean Walsh, *Philosophy and Model Theory** (OUP, 2018). This both explains technical results in model theory, and also explores the appeals to model theory in various branches of philosophy, particularly philosophy of mathematics, but in metaphysics more generally (recall 'Putnam's model-theoretic argument'), the philosophy of science, philosophical logic and more. So that's a very scattered literature that is being expounded, brought together, examined, inter-related, criticised and discussed. Button and Walsh don't pretend to be giving the last word on the many and varied topics they discuss; but they are offering us a very generous helping of first words and second thoughts. It's a large book because it is to a significant extent self-contained: model-theoretic notions get defined as needed, and many of the most philosophically significant results are proved.

The expositions of the technical stuff are usually exemplary (the authors have a good policy of shuffling some extended proofs into chapter appendices), and the philosophical discussion is done with vigour and a very engaging style. The breadth and depth of knowledge brought to the enterprise is remarkable. Philosophical readers of this Guide should find the book fascinating, then. And indeed, with judicious skimming/skipping (the signposting in the book is excellent), mathematicians with an interest in some foundational questions should find much of interest here too.

And that might already be about as far as many philosophers may want or need to go with model theory. Many mathematicians, however, will want to take the story about model theory rather further: so the story resumes in §??.

Postscript, mostly for mathematicians The blurb of Jonathan Kirby's recent *An Invitation to Model Theory* (CUP, 2019) describes the book's aim like this: "[T]raditional introductions to model theory assume a graduate-level background of the reader. In this innovative textbook, [the author] brings model theory to an undergraduate audience. The highlights of basic model theory are illustrated through examples from specific structures familiar from undergraduate mathematics . . ." Now, one thing that usually isn't familiar to undergraduate mathematicians is any serious logic: so, as you would expect, Kirby's book is an introduction to model theory that doesn't presuppose a first logic course. So he has to start with some rather speedy explanations about first-order languages and interpretations in structures (probably *too* speedy for a real newcomer).

The book *is* very clearly arranged – the early chapters would make very useful revision material. But though clearly written it all goes pretty briskly, with the first hundred pages covering much of what Manzano covers in her whole book. So I do think that some beginning readers would struggle with parts of this short book as a first introduction. If Kirby had slowed down a bit, giving further motivating classroom asides, adding a few extra illustrative examples of key concepts, etc., I think this could have made for a significantly more attractive text. As it stands, however, this *Invitation* doesn't supplant the previous recommendations. (For a little more about it, [see the book note here.](#))

Thanks to the efforts of the respective authors to write very accessibly, the suggested path through Chiswell & Hodges → (part of) Leary & Kristiansen → (excerpts from) Goldrei → Manzano is not at all a hard road to follow, yet we end up at least in the foothills of model theory. We can climb up to the same foothills by routes involving rather tougher scrambles, taking in some additional side-paths and new views along the way. Here are two suggestions for the more mathematical reader:

4. Shawn Hedman's *A First Course in Logic* (OUP, 2004) covers a surprising amount of model theory. Ch. 2 tells you about structures and about relations between structures. Ch. 4 starts with a nice presentation of a Henkin completeness proof, and then pauses (as Goldrei does) to fill in some background about infinite cardinals etc., before going on to prove the Löwenheim-Skolem theorems and compactness theorems. Then the rest of Ch. 4 and the next chapter covers more introductory model theory, though already touching on some topics beyond the scope of Manzano's book, and Hedman so far could serve as a rather tougher alternative to her treatment. (Then Ch. 6 takes the story on a lot further, quite a way beyond what I'd regard as 'entry level' model theory.) For more, see this [Book Note on Hedman.](#)
5. Peter Hinman's weighty *Fundamentals of Mathematical Logic* (A. K. Peters, 2005) is not for the faint-hearted, and I wouldn't at all recommend using this book as your guide in your *first* outing into this territory. But if you are mathematically minded and have already made a first foray along a gentler route, you could now try reading Ch. 1 – skipping material that is now familiar – and then carefully working through Ch. 2 and Ch. 3 (leaving the last two sections, along with a further chapter on model theory, for later). This should significantly deepen your knowledge of FOL, or at least of its semantic features, and of the beginnings of model theory. For more, see this [Book Note on Hinman.](#)

6 Computability and incompleteness

7 Beginning set theory

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