6.4 Slice categories

(a) Finally in this chapter we will look at another way of constructing new categories from old – or rather, we define a dual pair of constructions. This will take just a bit of care, and so provides a nice check on understanding.

So suppose that C is a category, and X a particular C-object. We are first going to define a new category C/X whose *objects* are the pairs $(A, f\colon A \to X)$ for any C-object A and any C-arrow f with that object A as source and the fixed object X as target.[1] And then there will be a dual construction, a new category X/C whose objects are all the pairs $(A, f\colon X \to A)$, where again A is an object and f an arrow in the original category C.

But why should we be interested in such constructions? Let's have a couple of very simple examples:

(1) Take an n-membered index set $I_n = \{c_1, c_2, c_3, \ldots, c_n\}$. Think of the members of I_n as 'colours'. Then a pair $(S, S \to I_n)$ gives us a set S whose members are coloured from that palette of n colours.

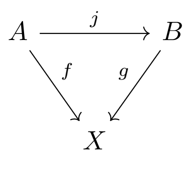
Hence we can, in particular, think of FinSet/I_n as the category of n-coloured finite sets, exactly the sort of structure that combinatorialists will be interested in.

(2) Pick a singleton set '1'. We have mentioned before that we can think of any element x of the set S as an arrow $\vec{x}\colon 1 \to S$.

So now think about a category $1/\mathsf{Set}$ whose objects are all the pairs $(S, \vec{x}\colon 1 \to S)$. So each such object of $1/\mathsf{Set}$ provides us with a set and then a distinguished element of that set; in other words, the object works as a pointed set. So, with the arrows of $1/\mathsf{Set}$ appropriately defined – we'll come to that in a moment! – $1/\mathsf{Set}$ will be (or, in some strong sense to be explained later, comes to the same as) the category Set_* of pointed sets.

True, pointed sets aren't very exciting. But pointed topological spaces are. And, with 1 now some one-point topological space, $1/\mathsf{Top}$ similarly gives us the category of pointed topological spaces.

(b) OK, now we have an albeit modest amount of motivation, let's ask what the *arrows* of a category C/X should be. C/X's objects are, to repeat, pairs of objects and arrows from C, as it might be $(A, f\colon A \to X)$ and $(B, g\colon B \to X)$. So if we are to derive C/X from data available in C, what can we use to construct an arrow from $(A, f\colon A \to X)$ to $(B, g\colon B \to X)$?



The obvious candidate is a C-arrow j from A to B which interacts appropriately with the arrows f and g, so we get a commuting diagram.

But now we need to be a bit careful. Shall we say (1) that C/X-arrow from $(A, f\colon A \to X)$ to $(B, g\colon B \to X)$ is simply

[1] Since an arrow has a unique source, on our definition of categories, we could without loss of information take the object-data of C/X to be simply the C-arrows $f\colon A \to X$. Many opt for this simpler definition for C/X-objects.

any $j\colon A \to B$ such that $f = g \circ j$? Or should we rather say (2) that a C/X-arrow is the whole commuting triangle, i.e. a triple (j, f, g) where $f = g \circ j$?

Here's a problem with option (1). Suppose that the sole C-arrow from A to A is 1_A. And consider the object $(A, f\colon A \to X)$ in C/X. On option (1) what would be that object's identity arrow in C/X? It will need to be a C-arrow $j\colon A \to A$ such that $f = f \circ j$. So it must be 1_A. But now suppose that there is another distinct arrow $f'\colon A \to X$ in C, giving us a distinct C/X-object $(A, f'\colon A \to C)$. What is the identity arrow on *this* object? By the same argument, it must again be 1_A. But then we'd have distinct objects with the same identity arrow on them, which is impossible by Theorem 9.

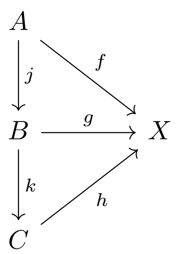
So the simpler option (1) for defining an arrow from f to g in C/X fails, and we need to give a somewhat more complicated story such as (2):

Definition 25. Let C be a category, and X be a C-object. Then the category C/X, the *slice category over* X, has the following data:

(1) The C/X-objects are all the pairs $(A, f\colon A \to X)$ for every C-object A and associated C-arrow $f\colon A \to X$.
(2) A C/X-arrow from $f\colon A \to X$ to $g\colon B \to X$ is a triple of C-arrows (j, f, g) where $f = g \circ j$ in C.
(3) The identity arrow in C/X on $(A, f\colon A \to X)$ is the triple of C-arrows $(1_A, f, f)$.
(4) Given C/X-arrows (j, f, g) and (k, g, h), their composition is $(k \circ j, f, h)$ (where $k \circ j$ is of course the composite arrow in C). △

By brute force, then, we've ensured that the identity arrows on $(A, f\colon A \to X)$ and $(A, f'\colon A \to X)$ are distinct if f and f' are.

Of course, we need to check that these data do together satisfy the axioms for constituting a category. So let's do that. In particular, we need to confirm that our definition of composition for C/X-arrows works.



If (j, f, g) is a C/X-arrow, then $g \circ j = f$ and the top triangle commutes in C. If (k, g, h) is C/X-arrow , then $h \circ k = g$ and the bottom triangle commutes in C. So pasting the triangles together, the whole resulting diagram commutes in C. Or in equations, we have $(h \circ k) \circ j = f$ in C, and therefore $h \circ (k \circ j) = f$. Hence $(k \circ j, f, h)$ really does count as an arrow in C/X from f to h, as we require.

The remaining checks to confirm C/X satisfies the axioms for being a category are then trivial.

(c) Now for the dual notion, namely the idea of a *co-slice category* X/C (or the slice category *under* X). As we said, the objects of this category are C-objects paired with C-arrows going in the opposite direction, i.e. they are of the form $(A, f\colon X \to A)$. Then the rest of the definition is exactly as you would predict given our explanation of duality: just go through the definition a slice category reversing arrows and the order of composition. It is a useful exercise to check that this works! And we will return to slice and co-slice categories in due course.